A postscript on degrees of freedom necessary for a turbulence calculation.

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In the preceding post, we discussed the new method that has been a proposed by Donzis and Sajeev which reduces the computational effort but still retains all the degrees of freedom [1]. In this method, all the degrees of freedom are divided at the time step $t=t_n$ into resolved modes, which are integrated forward in time to $ t=t_{n+1}$ using the NSE; and unresolved modes which are assumed not to change at that time step. The resolved modes are selected by a random sampling process which is carried out at each time step. In this way, the total number of degrees of freedom remains the same, but those actually being computed differ from one time step to another. This has the great advantage, compared to other methods, of apparently being able to correctly retain the initial value of the total energy. However, some qualification of this point needs to be made, as follows.

Consider the total energy $E(t_n)$ at time t_n, which is given by \begin{equation}E(t_n)=\frac{1}{2}\sum_{k=0}^N \langle u^2(k,t_n)\rangle, \end{equation}where N is the total number of wavenumber modes or degrees of freedom. (Note that in numerical simulation, the symbol N is used for the number of resolved modes in one direction, with N^3 being used for the total number of resolved modes being simulated.) For a forced, stationary simulation of the NSE, the total energy tends to fluctuate about a mean value as time goes on: see Figure 3 in Reference [2], despite the fact that in principle it should be constant. This behaviour is presumably due to rounding errors.

However, in the Donzis-Sajeev proposal, there may be an additional cause of fluctuation. At time $t=t_n$ they divide the total number of modes up into resolved modes, which are obtained by integrating the NSE forward in time to $t=t_{n+1}$, and are denoted by k_r; and unresolved modes which are kept at the same value and are denoted by k_u. The corresponding numbers of modes are $n(k_r)$ and $n(k_u)$, which add up to the total number $n(k_t)=N$. At the time $t=t_{n+1}$ the expression for the total energy should be decomposed into two parts: \begin{equation}E(t_{n+1})=\frac{1}{2}\sum_{k\in \{k_r\}}^{n(k_r)} \langle u^2(k,t_{n+1})\rangle +\frac{1}{2}\sum_{k\in \{k_u\}}^{n(k_u)} \langle u^2(k,t_n)\rangle . \end{equation} As the first term on the right belongs to the sub-ensemble of the resolved modes and the second term belongs to the sub-ensemble of the unresolved modes, there is no *a priori* reason that $E(t_n)$ should be equal to $E(t_{n+1})$. It would be interesting to see how the behaviour of the total energy would compare with that of the equivalent full simulation.

This will be my last post before the holidays. I hope to resume in January 2025.

[1] Diego Donzis and Shilpa Sajeev. Degrees of freedom and the dynamics of fully developed turbulence. Physical Review Fluids, 9:044605–1–11, 2024.
[2] W. D. McComb, A. Hunter, and C. Johnston. Conditional mode-elimination and the subgrid-modelling problem for isotropic turbulence. Phys. Fluids, 13:2030, 2001.