An assessment of Onsager's concept of scale invariance: 2

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For many years, arising from Onsager's observation in 1945 [1], the condition $\Pi_{max} = \varepsilon$ for a range of wavenumbers $k_{bot}\leq k \leq k_{top}$ has been seen as a criterion for the existence of an inertial range, and hence for the Kolmogorov $-5/3$ spectrum holding over that range. It has been a cornerstone of both statistical theories and direct numerical simulations of the Navier-Stokes equation. Thus, it is surprising that it has been subject to very little critical assessment. The picture established in many investigations is of the flux tending to a constant value, along with an increasing extent of the $k^{-5/3}$ spectrum, as the Reynolds number is increased.

However, a numerical investigation by Shanmugasundaram, which was based on the Local Energy Transfer (LET) theory [2], does not seem to support this conventional picture, which amounts to scale-invariance of the energy flux. For Taylor-Reynolds numbers $4.7\leq R_{\lambda} \leq 254$, the energy flux Π was found to take the peaked form predictable from the general behavioural arguments given in our previous post, with the peak occurring at $k=k_\star$; while, as the Reynolds number increases, the energy spectrum tends to the $-5/3$ form.

For the particular case of $R_{\lambda} = 254$, the energy spectrum in their Fig. 3 shows at least a decade of $k^{-5/3}$, as one would expect at that value of the Taylor-Reynolds number. Whereas, from their Fig. 6 we see that the energy flux, corresponding to the energy balance in the upper panel of the figure, takes the form of a peak, with Π_{max}/ε lying between 0.70 and 0.80, rather than the value of unity that Onsager suggested.

This would seem to be a good example of what prompted Kraichnan’s comment [3]: `Kolmogorov’s 1941 theory has achieved an embarrassment of success.’ In other words, despite the underlying conditions not apparently being satisfied, the $-5/3$ spectrum was still observed.

More recently, Meldi and Vassilicos [4] used the single-time EDQNM closure and, over a much greater range of Reynolds numbers, also found a small maximum in the inertial flux. They showed that the position of this maximum in wavenumber scaled on the Taylor micro length scale. Elementary calculus would then imply that the position of the single zero of the transfer spectrum also depended on the Taylor microscale and the authors confirmed that this was the case [5].

In the next post we shall argue that both these investigations were incorrect in using the flux $\Pi(k)$. Instead, they should have used the non-conservative part of the flux $\Pi^{-+}(k | k_{*})$ and the corresponding transfer spectrum: \begin{equation}T^{-+}(k|k_{*})=\int_{k_{*}}^\infty\,dj S(k,j) \quad \mbox{for} \quad 0\leq k \leq k_{*},\end{equation} as introduced by McComb [6] in the course of resolving the scale-invariance paradox.

References
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