An assessment of Onsager’s concept of scale invariance: 1

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We stated Onsager’s criterion [1] for an inertial range in terms of a scale-invariant flux as equation (4) in our post of 24 June 2024. In order to assess Onsager’s concept, we begin by considering the Lin equation in terms of the energy spectrum $E(k,t)$ and the transfer spectrum $T(k,t)$. We may write it in its well-known form: \begin{equation}\frac{d E(k,t)}{dt} = T(k,t) – 2\nu k^2E(k,t) \equiv T(k,t) – D(k,t),\label{lin}\end{equation}where $D(k,t)$ is the energy dissipation spectrum, as given by $D(k,t) = 2\nu k^{2}E(k,t)$. Here, we assume that there are no forces acting. We may also express the transfer spectrum in terms of its spectral density $S(k,j;t)$ thus: \begin{equation} T(k,t) = \int_0^\infty\, dj \,S(k,j;t),\quad \mbox{where} \quad S(k,j;t) = -S(j,k;t);\label{Tdef}\end{equation}and $S(k,j;t)$ contains the triple moment in wavenumber space: see [2]. Note that the antisymmetry of $S(k,j;t)$ under the interchange of k and j guarantees that conservation of energy is maintained in the form $\int_0^\infty\, dk T(k) = 0$. When we substitute for $T(k)$ in terms of $S(k,j;t)$, we obtain the second form of the Lin equation.

If we write it in this, its full form, the Lin equation, tells us that all the Fourier modes are coupled to each other. It is, in the language of physics, an example of the *many-body problem*. It is in fact highly non-local, as in principle it couples every mode to every other mode. A corollary of this is that it predicts an energy cascade. This can be deduced from the nonlinear term which couples all modes together plus the presence of the viscous term which is symmetry-breaking. If the viscous term were set equal to zero, then the coupled but inviscid equation would yield equipartition states.

We may consider the transfer of energy from wavenumbers less than κ to wavenumbers greater than κ. To do this, we integrate the terms of the Lin equation from $k=0$ to an arbitarily chosen $k=\kappa$, with the result:\begin{eqnarray}\frac{d}{dt}\int^{\kappa}_{0}dk E(k,t) & = &\int_{0}^{\kappa}dk\int^{\infty}_{0}dj S(k,j;t)-\int^{\kappa}_{0}dk D(k,t) \nonumber \\& = & \int_{0}^{\kappa}dk\int^{\infty}_{\kappa}dj S(k,j;t)-\int^{\kappa}_{0}dk D(k,t),\label{linint}\end{eqnarray}where the second form of the right hand side relies on the fact that the double integral over $ 0\leq k,j \leq \kappa $ vanishes due to the antisymmetry of $S(k,j;t)$. Evidently this equation tells us that the loss of energy from modes with $k\leq \kappa$ is due to transfer to modes with $k\geq \kappa$, as well as the direct loss to dissipation.

In order to consider the inertial transfer further, we first introduce the symbol $E_{\kappa}(t)$ for the amount of energy contained in modes $0\leq k \leq \kappa$, thus:\begin{equation}\frac{d}{dt}E_{\kappa}(t) = \frac{d}{dt}\int^{\kappa}_{0}dk E(k,t).\end{equation}We then introduce the symbol $\Pi(\kappa,t)$, defined by:\begin{equation}\Pi(\kappa,t) = \int^{\infty}_{\kappa}dk T(k,t) = -\int^{\kappa}_{0}dk T(k,t)\label{Pidef}\end{equation}which represents the flux through mode $k=\kappa$; and, in terms of the transfer spectral density,\begin{equation} \Pi(\kappa,t) = \int^{\infty}_{\kappa}dk\int^{\kappa}_{0}dj\,S(k,j;t) = -\int^{\kappa}_{0}dk\int^{\infty}_{\kappa}dj\, S(k,j;t).\end{equation}
Accordingly, we may write the partially integrated Lin equation (\ref{linint}) as:\begin{equation}\frac{d}{dt}E_{\kappa}(t) = -\Pi(\kappa,t) – \int^{\kappa}_{0}dk\,D(k,t).\end{equation}Thus, the energy

contained in modes $0\leq k \leq \kappa$ is lost partly to direct viscous dissipation and partly due to transfer to higher-wavenumber modes.This is the only concept of localness which is required for the Richardson-Kolmogorov picture in wavenumber space (K41).

This analysis holds for any wavenumber $k=\kappa$, but the most important case occurs at $\kappa =k_{*}$, which is the wavenumber where $T(k,t)$ has its single zero-crossing. As is well known, the zeros of the transfer spectrum are given by:\begin{equation}T(0,t) =0;\qquad T(k_{*},t)=0; \quad \mbox{and} \quad \lim_{k \to \infty} T(k,t)=0: \end{equation}while, by simple calculus, the behaviour of the energy flux is given by: \begin{equation}\Pi(0,t) =0;\qquad \Pi(k_{*},t)=\Pi_{max}; \quad \mbox{and} \quad \lim_{k \to \infty} \Pi(k,t)=0.\end{equation}
It follows from conservation of energy that the maximum value that Π_{max} can take is the rate of viscous dissipation, thus we have the general result:\begin{equation}\Pi_{max} \leq \varepsilon;\end{equation}where the equality applies if the *local* viscous dissipation can be neglected at $k=k_{*}$.

In our next post, we will take a more critical look at this criterion.

References.
[1] L. Onsager. The Distribution of Energy in Turbulence. Phys. Rev., 68:286, 1945.
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Onsager’s (1945) interpretation of Kolmogorov’s (1941a) theory: 4

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In his 1964 paper, Corrsin [1] explained Onsager’s theory in a more up to date notation. We will build on that treatment here, but we shall use an even more modern notation. In particular, we will use $E(k)$ for the energy spectrum and ε for the dissipation rate.

In general, energy flows from small wavenumbers to large, and Corrsin noted that Onsager envisaged this cascade as proceeding stepwise, with the sequence of wavenumbers involved taking the form of a geometric progression. He had chosen this to be wavenumber doubling at each step, with the implication that the step length was $\Delta k = k$. Arguably the amount of energy transferred at each step is: \begin{equation}\Delta E = \Delta k E(k)= kE(k)\end{equation}

Representing the flux of energy through wavenumber k by $\Pi(k)$, we may write an approximate expression for it as: \begin{equation}\Pi(k) \sim kE(k)/\tau(k),\end{equation} where $\tau(k)$ is an appropriate characteristic time for energy transfer through mode k.

We now concentrate on the inertial range and note that as the cascade is conservative and there is no significant loss of energy to viscous dissipation in this range, we may write: \begin{equation}d\Pi/dk =0, \end{equation} with the integral result: \begin{equation}\Pi = \varepsilon.\end{equation} This means that the energy flux is independent of wavenumber in the inertial range. Nowadays, this is referred to as `scale-invariance' of the energy flux in the inertial range and is widely used as a criterion for the presence of an inertial range. It is a very important concept and we shall subject it to critical scrutiny in later posts. For the moment, we will concentrate on showing how it leads to the $-5/3$ wavenumber spectrum in Onsager's theory.

In order to make progress, we introduce a characteristic time for energy transfer through mode k, which we denote by $\tau(k)$. From a simple dimensional argument, this is taken to be: \begin{equation}\tau(k)= \left[k^3E(k)\right]^{-1/2}.\end{equation} Then we substitute (5) into (4) and impose the invariance condition given by (4), to obtain for the energy spectrum: \begin{equation}E(k)=\alpha \varepsilon^{2/3}k^{-5/3},\end{equation} where the prefactor α is the well known Kolmogorov constant.

The introduction of the characteristic time $\tau(k)$ seems to be analogous to the renormalised inverse modal lifetime $\omega(k)$ which arises in the Edwards self-consistent field theory [2]. If we assume that one is the inverse of the other, then the substitution of the Kolmogorov spectrum, as given by (6), into (5) for the characteristic time yields: \begin{equation}\omega(k) = \frac{1}{\tau(k)}=\alpha^{1/2}\varepsilon^{1/3}k^{2/3},\end{equation} in agreement with the Edwards result.

It is worth noting that the power-law form of the energy spectrum is not so much an additional assumption, as it was in Kolmogorov's earlier theory, as a natural consequence of scale-invariance because it is a scale-invariant form.

References

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Onsager’s (1945) interpretation of Kolmogorov’s (1941a) theory: 3

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Returning to this topic after my holiday, I will focus on Onsager’s 1945 abstract [1]. This is brief to the point of being cryptic and requires exegesis, but we shall defer that to the next post. For the moment we will concentrate on its relationship to Kolmogorov K41A [2].

As we mentioned earlier, this fragment of Onsager’s work introduced the term `cascade’ as his interpretation of the Richardson-Kolmogorov picture of the nonlinear transfer of energy from large scales to small. Or, as he worked with wavenumber k, the energy cascade is from small wavenumbers to large, where it is terminated by the action of the viscosity. We shall not enlarge on that here, but merely note that he states that dimensional analysis leads to the expression for the spectral density \begin{equation}C(k) = \frac{E(k)}{4\pi k^2} = ({\mbox{universal factor}})\varepsilon^{2/3}k^{-11/3},\end{equation} where the `universal factor’ equals the Kolmogorov constant divided by 4π. The $-11/3$ power law may seem unfamiliar to most people who will be used to the $-5/3$ form, but in statistical theory it is usual to work with the spectral density $C(k)$.

Onsager also pointed out that the corresponding correlation function takes the form \begin{equation}f(r)=1-(\mbox{constant})r^{2/3}.\end{equation} The term `corresponding’ refers to Fourier transformation of equation (1). Note that, as well as modernising the notation, I have taken the correlation function to be the `longitudinal correlation function’. The relationship between $f(r)$ and the energy spectrum can be found as equation (2.91) in the book [3].

Bearing in mind that Kolmogorov worked with the structure functions, equation (2) is just his result with the factor $\varepsilon^{2/3}$ absorbed into the constant. In other words, we can derive Kolmogorov’s result for the second-order structure function by Fourier transforming Onsager’s result, and I shall argue in later posts that that is the fundamental derivation.

However, the argument works both ways, and we can argue that the $-5/3$ law for the spectrum can be derived trivially by Fourier transformation of Kolmogorov K41A for $S_2(r)$. Accordingly it is appropriate to refer to it as the Kolmogorov spectrum.

References
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