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In his 1964 paper, Corrsin [1] explained Onsager's theory in a more up to date notation. We will build on that treatment here, but we shall use an even more modern notation. In particular, we will use $E(k)$ for the energy spectrum and ε for the dissipation rate.

In general, energy flows from small wavenumbers to large, and Corrsin noted that Onsager envisaged this cascade as proceeding stepwise, with the sequence of wavenumbers involved taking the form of a geometric progression. He had chosen this to be wavenumber doubling at each step, with the implication that the step length was $\Delta k = k$. Arguably the amount of energy transferred at each step is: \begin{equation}\Delta E = \Delta k E(k)= kE(k)\end{equation}

Representing the flux of energy through wavenumber k by $\Pi(k)$, we may write an approximate expression for it as: \begin{equation}\Pi(k) \sim kE(k)/\tau(k),\end{equation} where $\tau(k)$ is an appropriate characteristic time for energy transfer through mode k.

We now concentrate on the inertial range and note that as the cascade is conservative and there is no significant loss of energy to viscous dissipation in this range, we may write: \begin{equation}d\Pi/dk =0, \end{equation} with the integral result: \begin{equation}\Pi = \varepsilon.\end{equation} This means that the energy flux is independent of wavenumber in the inertial range. Nowadays, this is referred to as `scale-invariance' of the energy flux in the inertial range and is widely used as a criterion for the presence of an inertial range. It is a very important concept and we shall subject it to critical scrutiny in later posts. For the moment, we will concentrate on showing how it leads to the $-5/3$ wavenumber spectrum in Onsager's theory.

In order to make progress, we introduce a characteristic time for energy transfer through mode k, which we denote by $\tau(k)$. From a simple dimensional argument, this is taken to be: \begin{equation}\tau(k)= \left[k^3E(k)\right]^{-1/2}.\end{equation} Then we substitute (5) into (4) and impose the invariance condition given by (4), to obtain for the energy spectrum: \begin{equation}E(k)=\alpha \varepsilon^{2/3}k^{-5/3},\end{equation} where the prefactor α is the well known Kolmogorov constant.

The introduction of the characteristic time $\tau(k)$ seems to be analogous to the renormalised inverse modal lifetime $\omega(k)$ which arises in the Edwards self-consistent field theory [2]. If we assume that one is the inverse of the other, then the substitution of the Kolmogorov spectrum, as given by (6), into (5) for the characteristic time yields: \begin{equation}\omega(k) = \frac{1}{\tau(k)}=\alpha^{1/2}\varepsilon^{1/3}k^{2/3},\end{equation} in agreement with the Edwards result.

It is worth noting that the power-law form of the energy spectrum is not so much an additional assumption, as it was in Kolmogorov's earlier theory, as a natural consequence of scale-invariance because it is a scale-invariant form.

References

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