Onsager’s (1945) interpretation of Kolmogorov’s (1941a) theory: 3

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Returning to this topic after my holiday, I will focus on Onsager’s 1945 abstract [1]. This is brief to the point of being cryptic and requires exegesis, but we shall defer that to the next post. For the moment we will concentrate on its relationship to Kolmogorov K41A [2].

As we mentioned earlier, this fragment of Onsager’s work introduced the term `cascade’ as his interpretation of the Richardson-Kolmogorov picture of the nonlinear transfer of energy from large scales to small. Or, as he worked with wavenumber k, the energy cascade is from small wavenumbers to large, where it is terminated by the action of the viscosity. We shall not enlarge on that here, but merely note that he states that dimensional analysis leads to the expression for the spectral density \begin{equation}C(k) = \frac{E(k)}{4\pi k^2} = ({\mbox{universal factor}})\varepsilon^{2/3}k^{-11/3},\end{equation} where the `universal factor’ equals the Kolmogorov constant divided by 4π. The $-11/3$ power law may seem unfamiliar to most people who will be used to the $-5/3$ form, but in statistical theory it is usual to work with the spectral density $C(k)$.

Onsager also pointed out that the corresponding correlation function takes the form \begin{equation}f(r)=1-(\mbox{constant})r^{2/3}.\end{equation} The term `corresponding’ refers to Fourier transformation of equation

(1). Note that, as well as modernising the notation, I have taken the correlation function to be the `longitudinal correlation function'. The relationship between $f(r)$ and the energy spectrum can be found as equation (2.91) in the book [3].

Bearing in mind that Kolmogorov worked with the structure functions, equation (2) is just his result with the factor $\varepsilon^{2/3}$ absorbed into the constant. In other words, we can derive Kolmogorov's result for the second-order structure function by Fourier transforming Onsager's result, and I shall argue in later posts that that is the fundamental derivation.

However, the argument works both ways, and we can argue that the $-5/3$ law for the spectrum can be derived trivially by Fourier transformation of Kolmogorov K41A for $S_2(r)$. Accordingly it is appropriate to refer to it as the Kolmogorov spectrum.

References

[1] L. Onsager. The Distribution of Energy in Turbulence. Phys. Rev., 68:286, 1945.
[2] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. C. R. Acad. Sci. URSS, 30:301, 1941.
[3] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.