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Before we discuss Onsager's contribution, it will be helpful to first make some observations
about Kolmogorov's derivation of the $r^{2/3}$ law for the second-order structure function in the inertial range of scales r [1]. This is sometimes referrred to as K41A and there are many treatments of it (e.g. see the book [2]) so we will not reproduce the details here. Instead, we will attempt to highlight aspects of it, when it is judged in the context of its pioneering status. What I mean by this will quickly emerge.

First of all, Kolmogorov envisaged the effect of the nonlinearity in terms of Richardson's pictorial view of eddies being created at ever smaller scales until the smallest eddies are damped by viscosity (see page 11 of the book [2]). Of course, Kolmogorov did not acknowledge this at the time, but he subsequently did so in his 1962 paper [3]. He then introduced the idea, familiar in statistical physics, that a stepwise stochastic process with decreasing scale, could lead to a range of scales in which average properties were independent of the conditions of formation. Among other things, he also introduced the ideas of local isotropy and local stationarity; but although these are of considerable importance, for a purely theoretical look at the problem, we will restrict our attention for the moment, to fields that are both isotropic and stationary.

Now, we are dealing with a model in which the basic entities – Richardson's `whirls' – are not well defined. Hence, different people have different ideas of what types of eddies may be involved. Accordingly, we should examine just how sensitive the measurement of the structure functions is to the form assumed for the eddies making up the turbulent field. Let us consider an experiment where an anemometer measures the instantaneous velocity $u(x; t)$ at points $x = 0$ and $x = r$, and we take the difference between these values $\Delta u(r; t) = u(r; t) – u(0; t)$. Trivially, this velocity difference $\Delta u(r; t)$ is a random variable, which fluctuates rapidly about a zero mean. Note that its mean is identically zero because its two constituent means are themselves zero.

Next, we consider the second-order moment of $\Delta u(r; t)$, which we obtain by ensemble
average. That is, labelling a single realization of the velocity field by a superscript i, we define the ensemble average by: \begin{equation}S_2(r) = \lim_{N\rightarrow \infty}
\frac{1}{N}\sum_{i=1}^N\left[\Delta^{\{I\}}u(r,t)\right]^2,\end{equation} where $S_2(r)$ is the second-order structure function of the velocity field. The value of N has to be determined empirically from a consideration of the limit.

It should be clear that the ensemble average involves a summation over many random phases with consequent cancellations. Thus, dogmatic arguments about the form of the `whirls', particularly those arguments which seek to refute the Richardson-Kolmogorov picture, have little place. To argue otherwise is to give undue weight to the one particular eddy structure that one is visualizing. Even so, when Kolmogorov made the assumption that, for values of r that are not too large, the structure function can only depend on the variables r, ε and ν, then that is just an assumption.

The next step, in obtaining the two-thirds law, is to argue

that, for very small values of r and/or very large values of the Reynolds number, the dependence on the viscosity can be dropped. This is an empirical matter, as it involves evaluating the appropriate limits, and
stems from the Navier-Stokes equation (NSE). While it is often said that Kolmogorov's result
does not stem from the NSE, in fact this particular step does.

The final step is to assume that the inertial-range structure function takes the form of a power law and to use dimensional analysis. Evidently, expressing the energy in terms of a rate of change of energy with time necessarily brings in the two-thirds power law. I emphasise that this step is an assumption because in a later post I shall argue that the power-law may have a different status in Onsager's theory.

I am publishing this post a day earlier than usual, as I will be going on holiday on Thursday. My hope is to carry on this series of posts in early June.

References

[1] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. C. R. Acad. Sci. URSS, 30:301, 1941.
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of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech.,
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