Two-time correlations and temporal spectra: the analysis by Tennekes [1].

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In this post we take a closer look at the analysis by Tennekes [1] in which he differed from the earlier analysis of Tennekes and Lumley [2] and concluded that large-scale sweeping is the determining factor in the decorrelation of the two-time correlation in the inertial range. As noted in my post of 27 April 2023, this leads (rather confusingly) to a `$-5/3$' power law for the Eulerian temporal spectrum, when the Kolmogorov form is actually $n=-2$. His starting point is equation (1) in [1], which may be written in our present notation as: \begin{equation}\frac{\partial u_1}{\partial t}=-\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right),\end{equation} and this is justified by assuming that Taylor's hypothesis of frozen convection applies.

The usual application of Taylor's hyopothesis is to situations where there is a mean or free stream velocity U_1, which is much larger than the turbulent velocity $\mathbf{u}(\mathbf{x},t)$. Then the changes in the velocity field with time at a fixed measuring point could be due to the passage of a frozen pattern of turbulent motion past that point. Hence the local time derivative at a point may be replaced by the convective derivative, thus: \begin{equation}\frac{\partial}{\partial t} \rightarrow -U_1\frac{\partial}{\partial x_1} \quad \mbox{if} \quad U_1 \gg u.\end{equation} Or in the context of spectra, \begin{equation}k_1 = \omega/U_1.\end{equation} A fuller discussion of this can be found in Section 2.6.5 of [3].

Thus (1) seems a rather extreme application of Taylor’s hypothesis. In fact we can write down an exact expression for ${\partial u_1}/{\partial t}$ by invoking the Navier-Stokes equation. This gives us \begin{equation}\frac{\partial u_1}{\partial t}=-\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right)-\frac{\partial p}{\partial x_1} + \nu \nabla^2 u_1,\end{equation} where p is the kinematic pressure and ν is the kinematic viscosity. Thus in using equation (1), Tennekes neglects both the pressure and the viscous terms. The latter may seem reasonable, as his main concern was with the inertial range, but it must be borne in mind that the subsequent analysis involves squaring and averaging both sides of equation (1) so the neglect of the viscous term may introduce significant error. However, the neglect of the pressure term is even more concerning, as this is a highly non-local term with the pressure being expressed in terms of integrals of functions of the velocity field over the entire system volume: see Section 2.1 of [3].

This analysis relies on imponderable assumptions about scale separation and statistical independence. Such ideas were discussed much later on, and rather more quantitatively, in the context of mode eliminations and large eddy simulation: see Chapter 8 in the book [4] for an account of this work. It is clear that the analysis by Tennekes has swept a great deal under the carpet. In contrast, the arguments given by Tennekes and Lumley [2] seem, to me at least, more confident and well justified than those given in [1]. In his conclusion, Tennekes remarked on the difference between the two analyses, stating that it was `embarrassing in a personal sense.’ Certainly both sets of arguments might repay closer study.

As a final point, he expresses the view that the implications of [1] support Kraichnan’s view that Lagrangian coordinates are more suited to statistical closure theories than the more usual Eulerian variety. However, it is worth pointing out that all the analyses that support such a view are valid (if at all) only for stationary turbulence, whereas all the numerical assessments of closure theories are restricted to freely decaying turbulence. I intend to go on working on this topic as time permits.

[1] H. Tennekes. Eulerian and Lagrangian time microscales in isotropic turbulence. J. Fluid Mech., 87:561, 1975.

[2] H. Tennekes and J. L. Lumley. A first course in turbulence. MIT Press. Cambridge, Mass., 1972.

[3] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press,
1990.

[4] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology,
Renormalization and Statistical Closures. Oxford University Press, 2014.

Two-time correlations and temporal spectra: the Lagrangian case.

Two-time correlations and temporal spectra: the Lagrangian case.
In my previous post on 27 April 2023, I promised to come back to the Lagrangian case. Over the years, I have taken the view that the discussion of the Lagrangian case along with the Eulerian case, which is the one that is of more practical importance, is an unnecessary complication. At the same time,

I have had to acknowledge that the application of these ideas to the assessment of statistical closure theories should take account of the fact that there are Lagrangian theories as well as Eulerian theories. However, there is an interesting point to be made when we compare the treatment in the book by Tennekes and Lumley [1] with the later analysis of Tennekes [2].

In the previous post, we only mentioned the discussion by Tennekes and Lumley [1] of the inertial range behaviour of the Eulerian spectrum. In fact they not only derive the inertial range form of the Lagrangian spectrum, and find it to be the same power law as the Eulerian case, but also obtain a relationship between the constants of proportionality in the two cases.

The crucial step in this work is the equivalence of the two correlations (see Section 8.5 of [1]), where the authors refer back to their discussion of Lagrangian forms in Section 7.1 (actually they incorrectly give this as 7.2). Following their notation, we represent the Lagrangian velocity of a fluid point by $V_{\alpha}(t)$ where $\alpha = 1, 2, \mbox{or}, 3$. Then, they assert that $\langle V_{\alpha}V_{\alpha}\rangle =\langle u_{\alpha} u_{\alpha}\rangle$, where u_{α} is of course the Eulerian velocity; leading on to their equation (8.5.3). This is the step that provides the basis for their assertion of the equivalence of the Eulerian and Lagrangian inertial range spectra.

However, the later work of Tennekes [2] leads to the Eulerian spectrum being different from the Lagrangian form, due to the supposed predominance of sweeping effects. This would seem to be an inconsistency and we will return to this in future posts when we examine the work of Tennekes more closely.

We close by pointing out that in our previous post we noted that the form of two-time correlation being studied in [1] was limited to stationary flows. This point was also made by Hinze

[3]: see equation (1.57), page 39 in the first edition. However, in discussing the motion of fluid points in Lagrangian coordinates, Tennekes and Lumley emphasise the need for both homogeneity and stationarity. So in effect this restriction has already been made. We also note that an alternative discussion of the original work by Lumley can be found in Section 12.2 of [4].

[1] H. Tennekes and J. L. Lumley. A first course in turbulence. MIT Press. Cambridge, Mass., 1972.
[2] H. Tennekes. Eulerian and Lagrangian time microscales in isotropic turbulence. J. Fluid Mech., 87:561, 1975.

[3] J. O. Hinze. Turbulence. McGraw-Hill, New York, 1st edition, 1959.

[4] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press,
1990.