Two-time correlations and temporal spectra: the analysis by Tennekes [1].

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In this post we take a closer look at the analysis by Tennekes [1] in which he differed from the earlier analysis of Tennekes and Lumley [2] and concluded that large-scale sweeping is the determining factor in the decorrelation of the two-time correlation in the inertial range. As noted in my post of 27 April 2023, this leads (rather confusingly) to a `$-5/3$' power law for the Eulerian temporal spectrum, when the Kolmogorov form is actually $n=-2$. His starting point is equation (1) in [1], which may be written in our present notation as: \begin{equation}\frac{\partial u_1}{\partial t}=-\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right),\end{equation} and this is justified by assuming that Taylor's hypothesis of frozen convection applies.

The usual application of Taylor's hyopothesis is to situations where there is a mean or free stream velocity U_1, which is much larger than the turbulent velocity $\mathbf{u}(\mathbf{x},t)$. Then the changes in the velocity field with time at a fixed measuring point could be due to the passage of a frozen pattern of turbulent motion past that point. Hence the local time derivative at a point may be replaced by the convective derivative, thus: \begin{equation}\frac{\partial}{\partial t} \rightarrow -U_1\frac{\partial}{\partial x_1} \quad \mbox{if} \quad U_1 \gg u.\end{equation} Or in the context of spectra, \begin{equation}k_1 = \omega/U_1.\end{equation} A fuller discussion of this can be found in Section 2.6.5 of [3].

Thus (1) seems a rather extreme application of Taylor’s hypothesis. In fact we can write down an exact expression for ${\partial u_1}/{\partial t}$ by invoking the Navier-Stokes equation. This gives us \begin{equation}\frac{\partial u_1}{\partial t}=-\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}\right)-\frac{\partial p}{\partial x_1} + \nu \nabla^2 u_1,\end{equation} where p is the kinematic pressure and ν is the kinematic viscosity. Thus in using equation (1), Tennekes neglects both the pressure and the viscous terms. The latter may seem reasonable, as his main concern was with the inertial range, but it must be borne in mind that the subsequent analysis involves squaring and averaging both sides of equation (1) so the neglect of the viscous term may introduce significant error. However, the neglect of the pressure term is even more concerning, as this is a highly non-local term with the pressure being expressed in terms of integrals of functions of the velocity field over the entire system volume: see Section 2.1 of [3].

This analysis relies on imponderable assumptions about scale separation and statistical independence. Such ideas were discussed much later on, and rather more quantitatively, in the context of mode eliminations and large eddy simulation: see Chapter 8 in the book [4] for an account of this work. It is clear that the analysis by Tennekes has swept a great deal under the carpet. In contrast, the arguments given by Tennekes and Lumley [2] seem, to me at least, more confident and well justified than those given in [1]. In his conclusion, Tennekes remarked on the difference between the two analyses, stating that it was `embarrassing in a personal sense.’ Certainly both sets of arguments might repay closer study.

As a final point, he expresses the view that the implications of [1] support Kraichnan’s view that Lagrangian coordinates are more suited to statistical closure theories than the more usual Eulerian variety. However, it is worth pointing out that all the analyses that support such a view are valid (if at all) only for stationary turbulence, whereas all the numerical assessments of closure theories are restricted to freely decaying turbulence. I intend to go on working on this topic as time permits.

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[3] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press,
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[4] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology,
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