

Mode elimination: taking the phases into account: 2

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In last week's post, we mentioned Saffman's criticism of models like Heisenberg's theory of the energy spectrum in terms of their failure to take the phases into account. In this post we explore this idea and try to elucidate this criticism a little further. We can take Heisenberg's model as representative and an introductory discussion of it can be found in Section 2.8.1 of reference [1]. It is also discussed in Batchelor's book, and he made the general comment about it: 'The notion that the small eddies act as an effective viscosity is plausible enough but does not seem a suitable description of the mutual action of eddies whose sizes are of the same order of magnitude.'

In other words, he is expressing the need for what later became known as '*scale separation*': see, for example Section 5.1.1 of the book [3]. (Note: in last week's post I incorrectly wrote *scale invariance* when I meant *scale separation*. This has now been corrected.)

This is an important observation, and it is related to the way in which the phases come into the problem; but Batchelor did not mention this particular aspect. However, if we wish to be precise about the concept of phase, then we must turn again to Batchelor's book [2], where on page 83 he remarked that the Fourier components of the velocity field are complex, and hence may be written, in the usual way, in terms of an amplitude and a phase. In other words, for any particular wavenumber and time, $u_{\alpha}(\mathbf{k}, t)$ is a complex number.

Accordingly, following Batchelor, and with a change of

notation, we may write:
$$u_{\alpha}(\mathbf{k}, t) = |u_{\alpha}(\mathbf{k}, t)| \exp\{i\theta_{\alpha}(\mathbf{k}, t)\},$$
 where $i = \sqrt{-1}$, θ_{α} is the phase, and the Cartesian index α takes the values $\alpha=1,2,3$.

Batchelor then discussed its general importance, remarking that: 'The exchanges of energy are dependent, in general, on the relationships between the phases of the different Fourier component as well as on their amplitudes, and it is in the elucidation of the average properties of the phase relations that the key to the determination of the energy spectrum during the decay lies.'

Little more has been said on this aspect of turbulence theory, apart perhaps Kraichnan's frequent use of the term 'phase mixing' in his many papers on the direct interaction family of closures: for further details see either of the books [1] or [3]. In the next post we will look more specifically at mode elimination to try to establish what the limitations of the process are.

[1] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.

[2] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 2nd edition, 1971.

[3] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.