What are the first and second laws of turbulence?

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Occasionally I still see references in the literature to the *Zeroth Law of Turbulence.* The existence of a zeroth law would seem to imply that there is at least a first law as well. But, so far as I know, there are no other laws of turbulence, and hence my question is purely rhetorical.

The so-called zeroth law is the fact the turbulent dissipation tends to a limit as the Reynolds number increases. Some people seem to be obsessed by the fact that this is equivalent to a finite dissipation limit as the viscosity tends to zero. Unfortunately, they become hypnotised by the zero viscosity and completely overlook the word \`limit'! This becomes translated into \`finite turbulent dissipation at zero viscosity' and is also referred to as the \`dissipation anomaly'. If this were true, then it certainly would be anomalous, to say the least. But it isn't true. Turbulent dissipation is ultimately, like all dissipation in fluid systems, the transformation of macroscopic kinetic energy into heat by the action of viscosity. No viscosity means no dissipation.

I do not wish to become hypnotised myself by this particular manifestation of folklore. I have written about it before in these blogs and will write about it again. Right now I wish to concentrate only on the oddity of the terminology: \`zeroth law'. Presumably it has been so named by analogy with the situation in thermodynamics, where the well-established first and second laws were later supplemented by both a third law and a zeroth law. The third law was part of the subject when I took my first degree but the zeroth law wasn't. It amounts essentially to a definition of temperature that provides a basis for its measurement. I suppose that it became thought to be so fundamental that it really ought to precede the existing first and second laws.

However, if that was the case, then surely it would be better to name it something like `The fundamental principle of thermodynamics'? The trouble with zeroth law is that zero means nothing. That is, when you don't have any of something, then you have zero.

It is a failure to recognise this that causes confusion about the calendar when a century changes. One needs to realize that there is no `year zero'. Everything is zero to begin with. Then we start counting seconds, minutes, days and 365 days later we have achieved one year which we denote by `1'. When we reach ten years, we have completed a decade, and we can label that year by `10', with zero fulfilling its mathematical significance by giving us a symbol for `10'. Thus the year 10 is the last year of the decade, the year 100 is last year of the century, and the year 1000 is the last year of the millennium. Thus Year 2000 is the last year of the second millennium and Year 2001 is the first year of the third millennium. (I hope that digression made sense!)

In my view, the use of the term `zeroth law' is lame in thermodynamics and doubly lame in turbulence, where we do not even have an agreed first law. It also reflects muddled thinking, based very largely on a failure to understand the mathematical concept of a limit, which ends up with the erroneous supposition that the infinite Reynolds number limit corresponds to the Euler equation. This amounts to a failure to recognize that the Euler equation throughout its entire life has been indomitably non-dissipative.

This will be my last blog of this year. I intend to resume posting in the new year. In the meantime, I hope that we shall all have a pleasant holiday.

The non-Markovian nature of turbulence 9: large-eddy simulation (LES) using closure theories.

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In this series of posts we have argued that the three pioneering theories of turbulence (due to Kraichnan, Edwards and Herring, respectively) are all Markovian with respect to wavenumber interactions. Thus, despite their many successful features, the ultimate failure of these theories to give the correct infinite-Reynolds number limit arises from the fact that they cannot reproduce the non-Markovian nature of fluid turbulence. In the immediately preceding post, we drew a distinction between the concept of a process being Markovian in its wavenumber interactions and the `almost-Markovian' nature of certain single-time theories, where the term `Markovian' refers to their development with time. In this final post in the series, we may shed some further light on these matters by considering the use of closures to calculate the subgrid viscosity for a large-eddy simulation.

This activity was initiated in 1976 by Kraichnan [1] who considered isotropic turbulence and based his approach on his own test-field model. In fact this publication led to quite a lot of activity by others, although this was generally based on the very similar EDQNM model (see the previous post).

The LES equations for isotropic turbulence can be formulated in wavenumber space by filtering the velocity field at some fixed cut-off wavenumber k_c. Then, for the explicit (resolved) wavenumbers $k\leq k_c$, we have the resolved velocity field $u^{<}(\mathbf{k,t})$; while the subgrid field takes the form $u^{>}(\mathbf{k,t})$ for $k_c\leq k$. Then substituting into the Navier-Stokes equations, we obtain separate equations for the low-k and high-k, ranges. However, the nonlinear term ensures that the two equations of motion are coupled together. This coupling of explicit and implicit modes is the subgrid modelling problem.

A detailed discussion of these matters may be found in Section 10.3 of the book [2], but here we only wish to sketch out some features of Kraichnan’s approach insofar as they bear on the earlier posts in this series. We may do this schematically in terms of the Lin equation, as follows. Evidentally, corresponding to the explicit modes of the velocity field, we may define an explicit modes energy spectral density $C^{<}(k,t)$, and correspondingly the filtered energy spectrum $E^{<}(k,t) = 4\pi k^2 C^{<}(k,t)$. Accordingly we may write the energy balance for the explicit modes as:
$$\begin{equation}\left(\frac{\partial}{\partial t} + 2\nu k^2 \right)E^{<}(k,t) = T^{<}(k,t) + T^{<>}(k,t),\end{equation}$$
where $T^{<}(k,t)$ is the transfer spectrum for the explicit modes and contains only couplings within these modes; whereas $T^{<>}(k,t)$ contains terms involving the implicit modes. Kraichnan proposed [1] that the second transfer term could be modelled in terms of an effective subgrid viscosity $\nu(k|k_c)$, such that $$\begin{equation}T^{<>}(k,t) \equiv T(k|k_c) =-2\nu(k|k_c)k^2 E^{<}(k,t),\end{equation}$$ where at the same time he introduced the parametric notation shown.

The point that we wish to highlight here is that in using $T(k|k_c)$ Kraichnan only took the output term into acccount.

In fact the input term, even if small, must be included. In fact there are circumstances where it is not small and in general $\nu(k_c|k)$ is not positive definite, nor should it be. Thus, an adherence to the Markovian point of view that underpinned the DIA and the other pioneering closures, leads to an incorrect result. A full discussion of this may by found in Section 10.3 of [2] and on page 394 Kraichnan's effective viscosity can be found as equation (10.17), while the corrected form with the input term of the transfer spectrum included may be found as a footnote on page 403 of the same reference.

As a corollary here, on page 392 of [2] I have noted that Kraichnan showed that his first lagrangian theory reduced to a Markovian form under certain circumstances. In the case of the LET theory, I know that it is non-Markovian but I had only assumed that was the case for all the Lagrangian theories. So, at least for the first one, it has been shown to be the case.

[1] R. H. Kraichnan. Eddy-viscosity in two and three dimensions. J. Atmos. Sci., 33:1521, 1976.
[2] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.

The non-Markovian nature of turbulence 8: Almost-Markovian models and theories

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Previously, in my post of 10 November 2022, I mentioned, purely for completeness, the work of Phythian [1] who presented a self-consistent theory that led to the DIA. The importance of this for Kraichnan was that it also led to a model representation of the DIA and in turn to the development of what he called `almost-Markovian’ theories. Some further discussion of this topic can be found in Section 6.3.2 of the book [2], but here we will concentrate on the general class of almost-Markovian models and theories. My concern here is to draw a distinction between their use of `Markovian’, which refers to evolution in time, and my use in this series of posts, which refers to interactions in wavenumber.

This class consists of the Eddy-damped, Quasi-normal, Markovian (EDQNM) model of Orszag in 1970 [3], the test-field model of Kraichnan in 1971 [4], the modified LET theory of McComb and Kiyani in 2005 [5], and the theory of Bos and Bertoglio in 2006 [6]. Here we follow the example of Kraichnan who described a theory which relied on a specific assumption that involved the introduction of an adjustable constant as a *model*. In order to illustrate what is going on in this kind of approach, I will discuss the EDQNM in some detail, as follows.

We begin with the quasi-normal expression for the transfer spectrum $T(k)$ from the Lin equation. This is found to be: \begin{eqnarray}T(k,t) & & =8\pi^2\int d^{3}j\,L\left(\mathbf{k},\mathbf{j}\right)\int_{0}^{t}ds\,R_0 \left(k;t,s\right)R_0\left(j;t,s\right)R_0\left(\left|\mathbf{ k}-\mathbf{j}\right|;t,s\right) \nonumber \\& \times &\left[C\left(j,s\right)C\left(\left|\mathbf{k}-\mathbf{j}\right|,s\right)- C\left(k,s\right)C\left(\left|\mathbf{k}-\mathbf{j}\right|,s\right)\right],\label{KWE2} \end{eqnarray} where the viscous response function is given by \[R_0(k;t,t’)=\exp[-\nu k^2 (t-t’)],\] and the coefficient $L(\mathbf{k,j})$ is defined as: \begin{equation}L(\mathbf{k,j}) = -2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\math bf{j})P_{\gamma\delta}(\mathbf{k-j}),\label{lkj1}\end{equation} and can be evaluated in terms of three scalar variables as \begin{equation}L(\mathbf{k,j}) = -\frac{\left[\mu\left(k^{2}+j^{2}\right)-kj\left(1+2\mu^{2}\right)\right]\left(1-\mu^{2}\right)kj}{k^{2}+j^{2}-2kj\mu},\label{lkj2}\end{equation} where μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$. For further discussion and details see Appendix C of the book [7].

Now Orszag argued that the failure of QN was basically due to the use of the viscous response function, when in fact one would expect that the turbulence interactions would contribute to the response function. Accordingly he proposed a modified response function: \begin{equation}R(k;t,t’)=\exp[-\omega(k)(t-t’)],\end{equation}where $\omega(k)$ is a renormalized inverse modal response time. One may note that this is now becoming the same form as that of the Edwards transfer spectrum, but that it is also *ad hoc* and thus there is the freedom to choose $\omega(k)$. After some experimentation using dimensional analysis, Orszag chose the form: \begin{equation}\omega(k)=\nu k^2 + g\left[\int_0^k dj j^2 E(j)\right]^{1/2},\end{equation} where the constant g is chosen to give the correct (i.e. experimental) result for the Kolmogorov spectrum. This is the *eddy damped* part of the model, so replacing R_0 by R gives us the EDQN.

Even with the introduction of the damping term, the EDQN model can still lead to negative spectra. This was cured by introducing the *Markovian* step with respect to time. This rested on the assumption that the characteristic time $[\omega(k) +\omega(j) + \omega(|\mathbf{k-j}|)]^{-1}$ is negligible compared to the evolution time of the products of covariances in the expression for $T(k)$. The equation for the transfer spectrum was Markovianised by replacing the time integral by a memory function $D(k,j;t)$, thus: \begin{equation}T(k,t) =8\pi^2\int d^{3}j\,L\left(\mathbf{k},\mathbf{j}\right) D(k,j;t)\left[C\left(j,s\right)C\left(\left|\mathbf{k}-\mathbf{j}\right|,s\right)-C\left(k,s\right)C\left(\left|\mathbf{k}-\mathbf{j}\right|,s\right)\right],\end{equation} where the memory function is given by \begin{equation}D(k,j;t)= \int_0^t ds \, \exp[\omega(k)+\omega(j)+\omega(|\mathbf{k-j}|)](t-s).\end{equation}This is now the EDQNM model.

When applied to the stationary case, this result for $T(k)$ is identical to the Edwards result, as given in the post of 3 November 2022; but there are crucial differences. The function $\omega(k)$ in the Edwards theory arises from a Markovian theory with respect to wavenumber interactions and is accordingly related to $T(k)$, thus giving the second equation of the closure. In contrast, the function $\omega(k)$ in EDQNM is fixed independently of the transfer spectrum by means of dimensional analysis and accordingly is not Markovian in the sense of the Edwards SCF. It is important to distinguish between the two kinds of Markovianisation.

In our next post, we will conclude this series of posts by discussing how these considerations affect the application of closures to large-eddy simulation.

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[6] W. J. T. Bos and J.-P. Bertoglio. A single-time, two-point closure based on fluid particle displacements. Phys. Fluids,

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