The non-Markovian nature of turbulence 9: large-eddy simulation (LES) using closure theories.

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In this series of posts we have argued that the three pioneering theories of turbulence (due to Kraichnan, Edwards and Herring, respectively) are all Markovian with respect to wavenumber interactions. Thus, despite their many successful features, the ultimate failure of these theories to give the correct infinite-Reynolds number limit arises from the fact that they cannot reproduce the non-Markovian nature of fluid turbulence. In the immediately preceding post, we drew a distinction between the concept of a process being Markovian in its wavenumber interactions and the `almost-Markovian' nature of certain single-time theories, where the term `Markovian' refers to their development with time. In this final post in the series, we may shed some further light on these matters by considering the use of closures to calculate the subgrid viscosity for a large-eddy simulation.

This activity was initiated in 1976 by Kraichnan [1] who considered isotropic turbulence and based his approach on his own test-field model. In fact this publication led to quite a lot of activity by others, although this was generally based on the very similar EDQNM model (see the previous post).

The LES equations for isotropic turbulence can be formulated in wavenumber space by filtering the velocity field at some fixed cut-off wavenumber k_c. Then, for the explicit (resolved) wavenumbers $k\leq k_c$, we have the resolved velocity field $u^{<}(\mathbf{k,t})$; while the subgrid field takes the form $u^{>}(\mathbf{k,t})$ for $k_c\leq k$. Then substituting into the Navier-Stokes equations, we obtain separate equations for the low-k and high-k, ranges. However, the nonlinear term ensures that the two equations of motion are coupled together. This coupling of explicit and implicit modes is the subgrid modelling problem.

A detailed discussion of these matters may be found in Section 10.3 of the book [2], but here we only wish to sketch out some features of Kraichnan's approach insofar as they bear on the earlier posts in this series. We may do this schematically in terms of the Lin equation, as follows. Evidentally, corresponding to the explicit modes of the velocity field, we may define an explicit modes energy spectral density $C^{<}(k,t)$, and correspondingly the filtered energy spectrum $E^{<}(k,t) = 4\pi k^2 C^{<}(k,t)$. Accordingly we may write the energy balance for the explicit modes as: \begin{equation}\left(\frac{\partial}{\partial t} + 2\nu k^2 \right)E^{<}(k,t) = T^{<}(k,t) + T^{<>}(k,t),\end{equation} where $T^{<}(k,t)$ is the transfer spectrum for the explicit modes and contains only couplings within these modes; whereas $T^{<>}(k,t)$ contains terms involving the implicit modes. Kraichnan proposed [1] that the second transfer term could be modelled in terms of an effective subgrid viscosity $\nu(k|k_c)$, such that \begin{equation}T^{<>}(k,t) \equiv T(k|k_c) =-2\nu(k|k_c)k^2 E^{<}(k,t),\end{equation} where at the same time he introduced the parametric notation shown.

The point that we wish to highlight here is that in using $T(k|k_c)$ Kraichnan only took the output term into acccount. In fact the input term, even if small, must be included. In fact there are circumstances where it is not small and in general $\nu(k_c|k)$ is not positive definite, nor should it be. Thus, an adherence to the Markovian point of view that underpinned the DIA and the other pioneering closures, leads to an incorrect result. A full discussion of this may by found in Section 10.3 of [2] and on page 394 Kraichnan's effective viscosity can be found as equation (10.17), while the corrected form with the input term of the transfer spectrum included may be found as a footnote on page 403 of the same reference.

As a corollary here, on page 392 of [2] I have noted that Kraichnan showed that his first lagrangian theory reduced to a Markovian form under certain circumstances. In the case of the LET theory, I know that it is non-Markovian but I had only assumed that was the case for all the Lagrangian theories. So, at least for the first one, it has been shown to be the case.

[1] R. H. Kraichnan. Eddy-viscosity in two and three dimensions. J. Atmos. Sci., 33:1521, 1976.
[2] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.