The non-Markovian nature of turbulence 3: the Master Equation.

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In the previous post we established that the ‘loss’ term in the transport equation depends on the number of particles in the state currently being studied. This followed straightforwardly from our consideration of hard-sphere collisions. Now we want to establish that this is a general consequence of a Markov process, of which the problem of N hard spheres is a particular example.

We follow the treatment given in pages 162-163 of the book [1] and consider the case of Brownian motion, as this is relevant to the Edwards self-consistent field theory of turbulence. We again consider a multipoint joint probability distribution and now consider a continuous variable X which takes on specific values x_1 at time t_1, x_2 at time t_2, and in general x_n at time t_n; thus; \[f_n(x_1,t_1; x_2,t_2; \dots x_n,t_n).\] We then introduce the conditional probability density: \[p(x_1,t_1|x_2,t_2),\] which is the probability density that $X=x_2$ at $t=t_2$, given that X had the value $X=x_1$ when $t=t_1\leq t_2$. It is defined by the identity: \begin{equation}f_1(x_1,t_1)p(x_1,t_1|x_2,t_2)=f_2(x_1,t_1; x_2,t_2).\label{pdef}\end{equation}

From this equation (see [1]), we can obtain a general relationship between the single-particle probabilities at different times as: \begin{equation}f_1(x_2,t_2)=\int p(x_1,t_1|x_2,t_2)f_1(x_1,t_1)dx_1. \label{pprop}\end{equation}

Next we formally introduce the concept of a Markov process. We now define this in terms of the conditional probabilities. If: \begin{equation}p(x_1,t_1;x_2,t_2; \dots n_{n-1},t_{n-1}|x_n,t_n)=p(n_{n-1},t_{n-1}|x_n,t_n),\label{markdef}\end{equation}then the current step depends *only* on the immediately preceding step, and not on any other preceding steps. Under these circumstances the process is said to be Markovian.

It follows that the entire hierarchy of probability distributions can be constructed from the single-particle distribution $f_1(x_1,t_1)$ and the transition probability $p(x_1,t_1|x_2,t_2)$. The latter quantity can be shown to satisfy the Chapman-Kolmogorov equation: \begin{equation}p(x_1,t_1|x_3,t_3)=\int p(x_1,t_1|x_2,t_2)p(x_2,t_2|x_3,t_3) dx_2,\label{ck}\end{equation} indicating the transitive property of the transition probability.

It is of interest to consider two specific cases.

First, for a chain which has small steps between events, the integral relation (\ref{ck}) can be turned into a differential equation by expanding the time dependences to first order in Taylor series. Putting $f_1 = f$ for simplicity, we may obtain: \begin{equation}\frac{\partial f(x_2,t_2)}{\partial t}=\int\, dx_1\left\{W(x_1,x_2)f(x_1,t)-W(x_2,x_1)f(x_2,t)\right\}, \label{me}\end{equation} where $W(x_1,x_2)$ is the rate per unit time at which transitions from state x_1 to state x_2 take place. This is known as the *master equation*.

Secondly, if X is a continuum variable, we can further derive the Fokker-Planck equation as: \begin{equation} \frac{\partial f(x,t)}{\partial t}= \frac{\partial [A(x)f(x,t)]}{\partial x} + \frac{\frac{1}{2}\partial^2[B(x)f(x,t)]}{\partial x^2}. \label{fp}\end{equation} This equation describes a random walk with diffusivity $B(x)$ and friction damping $A(x)$. A discussion of this equation as applied to Brownian motion may be found on pages 163-164 of [1] but we will not pursue that here.

In the next post we will discuss the Edwards theory of turbulence (and by extension the other pioneering theories of Kraichnan and of Herring) in the context of the present work.

[1] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press 1990.