The non-Markovian nature of turbulence 2: The influence of the kinetic equation of statistical physics.

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The pioneering theories of turbulence which we discussed in the previous post were formulated by theoretical physicists who were undoubtedly influenced by their background in statistical physics. In this post we will look at one particular aspect of this, the Boltzmann equation; and in the next post we will consider the idea of Markov processes more explicitly.

For many people, a Markov process is associated with the concept of a random walk, where the current step depends only on the previous one and memory effects are unimportant. However, for our present purposes, we will need the more general formulation as developed in the context of the kinetic equations of statistical mechanics. A reasonably full treatment of this topic may be found in chapter four of the book [1], along with some more general references. Here we will only need a brief summary, as follows.

We begin with a system of N particles satisfying Hamilton's equations (e.g. a gas in a box). We take this to be spatially homogeneous, so that distributions depend only on velocities and not on positions. Conservation of probability implies the exact Liouville equation for the N-particle distribution function f_N, but in practice we would like to have the single-particle distribution $f_1(u,t)$. If we integrate out independent variables progressively, this leads to a statistical hierarchy of governing equations, in which each *reduced distribution* depends on the previous member of the hierarchy: a closure problem!

The hierarchy terminates with an equation for the single-point distribution f_1 in terms of the two-particle distribution f_2. This is known as *the kinetic equation*. The kinetic equation for $f_1(x,u,t)$ may be written as: \begin{equation}\frac{\partial f_1}{\partial t} + (u.\nabla) f_1 =\{\mbox{Term involving}\, f_2\}, \end{equation} where x is the position of a single particle, u is its velocity, and ∇ is the gradient operator with respect to the variable x. If we follow Boltzmann and model the gas molecules as hard spheres, then we can assume that the right hand side of the equation is entirely due to collisions. Accordingly, we may write the kinetic equation as: \begin{equation}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{collisions}, \end{equation} where the convective term vanishes because of the previously assumed homogeneity. Also, we drop the subscript `1’ as we will only be working with the single-particle distribution.

Now let us consider the basic physics of the collisions. We assume that three-body collisions are unlikely and restrict our attention to the two-body case. Assume we have a collision in which a particle with velocity u collides with another particle moving with velocity v, resulting in two particles with velocities $u’$ and $v’$. Evidently this represents a *loss* of one particle from the set of particles with velocity u. Conversely, the inverse two-body collision can result in the *gain* of one particle to the state u. Hence we may interpret the right hand side of (2) as: \begin{equation}\left(\frac{\partial f}{\partial t}\right)_{collisions} = \mbox{Rate of gain to state}\,u\,-\mbox{Rate of loss from state}\,u. \end{equation}

The right hand side can be calculated using elementary scattering theory, along with the assumption of *molecular chaos* or *stossahlansatz,* in the form $f_2=f_1f_1$; with the result that equation (1) becomes: \begin{equation} \frac{\partial f(u,t)}{\partial t} = \int dv \int d\,\omega \,\sigma_d\,|u-v| \{f(u’,t)f(v’,t) – f(v,t)f(u,t)\},\end{equation} where σ_d is the differential scattering cross-section, the integral with respect to ω is over scattering angles, and the integral with respect to v stands for integration over all dummy velocity variables.

This is the Boltzmann equation and its key feature from our present point of view is that the rate of loss of particles from the state u depends on the number in that state, as given by $f(u,t)$. We will develop this further in the next post as being a general characteristic of Markovian theories. Of course the present treatment is rather sketchy, but a pedagogic discussion can be found in the book [2], which is free to download from Bookboon.com.

[1] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press 1990.
[2] W. David McComb. Study Notes for Statistical Physics: A concise, unified overview of the subject. Bookboon, 2014.