Turbulence renormalization and the Euler equation: 3

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In the previous post we saw that the mean-field and self-consistent assumptions/approximations are separate operations, although often referred to in the literature as if they could be used interchangeably. We also saw that the screened potential in a cloud of electrons could be interpreted as a Coulomb potential due to a renormalized charge. This type of interpretation was not immediately obvious for the magnetic case and indeed a much more elaborate statistical field theoretic approach would be needed to identify an analogous procedure in this case. It will be helpful to keep these thoughts in mind as we consider the theoretical approach to turbulence by Kraichnan in his DIA theory [1] in 1959. The other two key theories we shall consider are the diagrammatic method of Wyld [2] and the self-consistent field method of Edwards [3]. In what follows, we will adopt a simplified notation. Fuller details may be found in the books [4] or [5].

Kraichnan considered an infinitesimal fluctuation $\delta f(k,t)$ in the driving forces producing a fluctuation in the velocity field $\delta u(k,t)$. He then differentiated the NSE with respect to f to obtain a governing equation for δu, with exact solution: $$\delta u(k,t) = \int_{-\infty}^t \hat{G}(k;t,t') \delta f(k,t')dt',$$ where $\hat{G}$ is the infinitesimal response function. In this work Kraichnan made use of a mean-field assumption, viz. $$\langle \hat{G}(t,t')u(t)u(t')\rangle = \langle\hat{G}(t,t')\rangle \langle u(t)u(t')\rangle = G(t,t') \langle u(t)u(t')\rangle,$$ where G is the response function that is used for the subsequent perturbation theory.

For perturbation theory, a book-keeping parameter λ

(ultimately set equal to unity) is introduced to mutliply the nonlinear term and G is expanded in powers of λ, thus: \[G(t,t’)= G_0(t,t’)+\lambda G_1(t,t’) + \lambda^2 G_2(t,t’) + \dots\] For the zero-order term, we set the nonlinear term in the Navier-Stokes equation (NSE) equal to zero and the exact solution is: \[G_0(k,t-t’) = \exp[-\nu k^2(t-t’)],\quad\mbox{for}\quad t\geq t’;\] where we have now introduced stationarity. This is the viscous response function. So the technique is to calculate an approximation to the exact response function by means of partial summations of the perturbation series to all orders. This can be thought of as renormalizing the viscosity and that interpretation emerges more clearly in the diagrammatic method of Wyld [2].

The work of Wyld is a very straightforward analysis of the closure problem using conventional perturbation theory and a field-theoretic approach. It has received criticism and comment over the years but the underlying problems are procedural and are readily addressed [6]. From our point of view the pedagogic aspects of his formalism are attractive and it is beyond dispute that at second-order of renormalized perturbation theory his results verify those of Kraichnan. This is an important point as Wyld’s method does not involve a mean-field approximation.

At this stage it is clear that these two approaches cannot be directly applied to the Euler equation as there is no viscosity, and indeed the idea of forcing it would raise questions which we will not explore here. The interesting point here is that the Edwards self-consistent method does not rely explicitly on viscosity; nor, in the absence of viscosity, does it require stirring forces. Essentially it involves a self-consistent solution of the Liouville equation for the probability distribution of the velocities and, as it was applied to the forced NSE, it actually does involve both viscosity and stirring. Indeed it is known to be cognate with both the Kraichnan and the Wyld theories [4], [5]. Hence, like them it can be interpreted in terms of a renormalization of the viscosity.

These three theories, and other related theories, are all Markovian with respect to wavenumber (as opposed to time). The exception is the Local Energy Transfer (LET) theory [7], which does not divide the nonlinear energy transfer spectrum into input and output parts. Recently it has been found that the application of the Edwards self-consistent field method to the case of two-time correlations leads to a non-Markovian (in wavenumber) theory which has the response function R(t,t') determined by:$$R(t,t') = \left\langle u(t)\tilde{f}(t')\right\rangle_{0},$$ where $\tilde{f}(t)$ is a quasi-entropic force derived from the base distribution and the subscript 0 denotes an average against that distribution. As pointed out in [8], the tilde distinguishes the quasi-entropic force from the stirring force f. Edwards showed that $\langle uf\rangle$ was the rate of doing work by the stirring forces on the velocity field, whereas the new quantity $\langle u\tilde{f}\rangle$ determines the two-time response. It would seem that the LET theory can be applied directly to the Euler equation and this is something I hope to report on in the near future.

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