# Alternative formulations for statistical theories: 2.

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Carrying on from my previous post, I thought it would be interesting to look at the effect of the different formulations on statistical closure theories. In order to keep matters as simple as possible, I am restricting my attention to single-time theories and their forms for the transfer spectrum $T(k,t)$ as it occurs in the Lin equation (see page 56 in [1]). For instance, the form for this due to Edwards [2] may be written in terms of the spectral energy density $C(k,t)$ (or spectral covariance) as: \begin{equation}T(k,t) = 4\pi k^{2}\int d^{3}j L(k,j,|\mathbf{k}-\mathbf{j}|)D(k,j,|\mathbf{k}-\mathbf{j}|)C( |\mathbf{k}-\mathbf{j}|,t)[C(j,t)-C(k,t)],\end{equation}where \begin{equation}D(k,j,|\mathbf{k}-\mathbf{j}|) = \frac{1}{\omega(k,t)+\omega(j,t)+\omega(|\mathbf{k}-\mathbf{j}|,t)},\end{equation}and $\omega(k,t)$ is the inverse modal response time. The geometric factor $L(\mathbf{k},\mathbf{j})$ is given by:\begin{equation}L(\mathbf{k},\mathbf{j}) = [\mu(k^{2}+j^{2})-kj(1+2\mu^{2})]\frac{(1-\mu^{2})kj}{k^{2}+j^{2}-2kj\mu},\end{equation} and can be seen by inspection to have the symmetry:\begin{equation}L(\mathbf{k},\mathbf{j}) = L(\mathbf{j,}\mathbf{k}).\end{equation}From this it follows, again by inspection, that the integral of the transfer spectrum vanishes, as it must to conserve energy.

Edwards derived this as a self-consistent mean-field solution to the Liouville equation that is associated with the Navier-Stokes equation, and specialised it to the stationary case. Later Orszag [3] derived a similar form by modifying the quasi-normality theory to obtain a closure called the eddydamped quasi-normality markovian (or EDQNM) model. Although physically motivated, this was an *ad hoc* procedure and involved an adjustable constant. For this reason it is strictly regarded as a *model* rather than a *theory*. As this closure is much used for practical applications, we write in terms of the energy spectrum $E(k,t)=4\pi k^2 C(k,t)$ as:

$$T(k,t) = \int _{p+q=k} D(k,p,q)(xy+z^{3}) E(q,t)[E(p,t)pk^{2}-E(k,t)p^{3}]\frac{dpdq}{pq},$$

where

$$D(k,p,q) = \frac{1}{\eta(k,t)+\eta(p,t)+\eta(q,t)},$$

and $\eta(k,t)$ is the inverse modal response time (equivalent to $\omega(k,t)$ in the Edwards theory, but determined in a different way). Also $(xy+z^{3})$ is a geometric factor, where $x$, $y$ and $z$ are the cosines of the angles of the triangle subtended, respectively, by $k$, $p$ and $q$.

My point here is that Orszag, like many others, followed Kraichnan rather than Edwards and it is clear that you cannot deduce the conservation properties of this formulation by inspection. I should emphasise that the formulation can be shown to be conservative. But it is, in my opinion, much more demanding and complicated than the Edwards form, as I found out when beginning my postgraduate research and I felt obliged to plough my way through it. At one point, Kraichnan acknowledged a personal communication from someone who had drawn his attention to an obscure trigonometrical identity which had proved crucial for his method. Ultimately I found the same identity in one of my old school textbooks [5]. The authors, both masters at Harrow School, had shown some prescience, as they noted that this identity was useful for applications!

During the first part of my research, I had to evaluate integrals which relied on the cancellation of pairs of terms which were separately divergent at the origin in wavenumber. At the time I felt that Kraichnan’s way of handling the three scalar wavenumbers would have been helpful, but I managed it nonetheless in the Edwards formulation. Later on I was to find out, as mentioned in the previous blog, that there were in fact snags to Kraichnan's method too.

In 1990 [4] I wrote about the widespread use of EDQNM in applications. What was true then is probably much more the case today. It seems a pity that someone does not break ranks and employ this useful model closure in the Edwards formulation, rather than make *ad hoc* corrections afterwards for the case of wavenumber triangles with one very small side.

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