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Carrying on from my previous post, I thought it would be interesting to look at the effect of the different formulations on statistical closure theories. In order to keep matters as simple as possible, I am restricting my attention to single-time theories and their forms for the transfer spectrum $T(k,t)$ as it occurs in the Lin equation (see page 56 in [1]). For instance, the form for this due to Edwards [2] may be written in terms of the spectral energy density $C(k,t)$ (or spectral covariance) as: \begin{equation}T(k,t) = 4\pi k^{2}\int d^{3}j L(k,j,|\mathbf{k}-\mathbf{j}|)D(k,j,|\mathbf{k}-\mathbf{j}|)C(|\mathbf{k}-\mathbf{j}|,t)[C(j,t)-C(k,t)],\end{equation}where \begin{equation}D(k,j,|\mathbf{k}-\mathbf{j}|) = \frac{1}{\omega(k,t)+\omega(j,t)+\omega(|\mathbf{k}-\mathbf{j}|,t)},\end{equation}and $\omega(k,t)$ is the inverse modal response time. The geometric factor $L(\mathbf{k},\mathbf{j})$ is given by:\begin{equation}L(\mathbf{k},\mathbf{j}) = [\mu(k^{2}+j^{2})-kj(1+2\mu^{2})]\frac{(1-\mu^{2})kj}{k^{2}+j^{2}-2kj\mu},\end{equation} and can be seen by inspection to have the symmetry:\begin{equation}L(\mathbf{k},\mathbf{j}) = L(\mathbf{j,}\mathbf{k}).\end{equation}From this it follows, again by inspection, that the integral of the transfer spectrum vanishes, as it must to conserve energy.

Edwards derived this as a self-consistent mean-field solution to the Liouville equation that is associated with the Navier-Stokes equation, and specialised it to the stationary case. Later Orszag [3] derived a similar form by modifying the quasi-normality theory to obtain a closure called the eddydamped quasi-normality markovian (or EDQNM) model. Although physically motivated, this was an *ad hoc* procedure and involved an adjustable constant. For this reason it is strictly regarded as a *model* rather than a *theory*. As this closure is much used for practical applications, we write in terms of the energy spectrum $E(k,t)=4\pi k^2 C(k,t)$ as:

$$T(k,t) = \int _{p+q=k} D(k,p,q)(xy+z^{3}) E(q,t)[E(p,t)pk^{2}-E(k,t)p^{3}]\frac{dpdq}{pq},$$

where

$$D(k,p,q) = \frac{1}{\eta(k,t)+\eta(p,t)+\eta(q,t)},$$

and $\eta(k,t)$ is the inverse modal response time (equivalent to $\omega(k,t)$ in the Edwards theory, but determined in a different way). Also $(xy+z^{3})$ is a geometric factor, where x, y and z are the cosines of the angles of the triangle subtended, respectively, by k, p and q.

My point here is that Orszag, like many others, followed Kraichnan rather than Edwards and it is clear that you cannot deduce the conservation properties of this formulation by inspection. I should emphasise that the formulation can be shown to be conservative. But it is, in my opinion, much more demanding and complicated than the Edwards form, as I found out when beginning my postgraduate research and I felt obliged to plough my way through it. At one point, Kraichnan acknowledged a personal communication from someone who had drawn his attention to an obscure trigonometrical identity which had proved crucial for his method. Ultimately I found the same identity in one of my old school textbooks [5]. The authors, both masters at Harrow School, had shown some prescience, as they noted that this identity was useful for applications!

During the first part of my research, I had to evaluate integrals which relied on the cancellation of pairs of terms which were separately divergent at the origin in wavenumber. At the time I felt that Kraichnan’s way of handling the three scalar wavenumbers would have been helpful, but I managed it nonetheless in the Edwards formulation. Later on I was to find out, as mentioned in the previous blog, that there were in fact snags to Kraichnan's method too.

In 1990 [4] I wrote about the widespread use of EDQNM in applications. What was true then is probably much more the case today. It seems a pity that someone does not break ranks and employ this useful model closure in the Edwards formulation, rather than make *ad hoc* corrections afterwards for the case of wavenumber triangles with one very small side.

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