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In the spectral representation of turbulence it is well known that interactions in wavenumber space involve triads of wave vectors, with the members of each triad combining to form a triangle. It is perhaps less well known that the way in which this constraint is handled can have practical consequences. This was brought home to me in 1984, when we published our first calculations of the Local Energy Transfer (LET) theory [1].

Our goal was to compare the LET predictions of freely decaying isotropic turbulence with those of Kraichnan's DIA, as first reported in 1964 [2]. With this in mind, we set out to calculate both DIA and LET under identical conditions; and also to compare out calculations of DIA with those of Kraichnan, in order to provide a benchmark. We applied the Edwards formulation [3] of the equations to both theories; but, apart from that, in order to ensure strict comparability we used exactly the same numerical methods as Kraichnan. Also, three of our initial spectral forms were the same as his, although we also introduced a fourth form to meet the suggestions of experimentalists when comparing with experimental results.

Reference should be made to [1] for details, but predictions of both theories were in line with experimental and numerical results in the field, with LET tending to give greater rates of energy transfer (and higher values of the evolved skewness factor) than DIA, which was assumed to be connected with its compatibility with the Kolmogorov spectrum. However, our calculation of the DIA value of the skewness was about 4% larger than Herring and Kraichnan found [4], which could only be explained by the different mathematical formulation.

Let us consider the two different ways of handling the wavenumber constraint, as follows.

Kraichnan’s notation involved the three wave vectors $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$; and used the identity: \begin{equation}\int d^3p\int d^3q \,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})f(k,p,q)=\int_{p+q=k}dpdq\frac{2\pi pq}{k}f(k,p,q), \end{equation} where the constraint is expressed by the Dirac delta function and $ f(k,p,q)$ is some relevant function. Note that the domain of integration is in the (p,q) plane, such that the condition $p+q=k$ is always satisfied.

Edwards [3] used a more conventional notation of $\mathbf{k}$, $\mathbf{j}$, and $\mathbf{l}$; and followed a more conventional route of simply integrating over one of the dummy wave vectors in order to eliminate the delta function, thus: \begin{equation}\int d^3j\int d^3l \,\delta(\mathbf{k}-\mathbf{j}-\mathbf{l})f(k,j,l)=\int_0^\infty 2\pi j^2 dj\int_{-1}^{1}d\mu f(k,j,|\mathbf{k}-\mathbf{j}|),\end{equation}where $\mu = \cos \theta_{kj}$ and θ_{kj} is the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$.

Of course the two formulations are mathematically equivalent. Where differences arise is in the way they handle rounding and truncation errors in numerical procedures. It was pointed out by Kraichnan [2], that corrections had to be made when triangles took the extreme form of having one side very much smaller than the other two. If this problem can lead to an error of about 4%, then it is worth investigating further. I will enlarge on this matter in my next post.

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