From minus five thirds in wavenumber to plus two-thirds in real space.

From $k^{-5/3}$ to $x^{2/3}$.

From time to time, I have remarked that all the controversy about Kolmogorov’s (1941) theory arises because his real-space derivation is rather imprecise. A rigorous derivation relies on a wavenumber-space treatment; and then, in principle, one could derive the two-thirds law for the second-order structure function from Fourier transformation of the minus five-thirds law for the energy spectrum. However, the fractional powers can seem rather daunting and when I was starting out I was fortunate to find a neat way of dealing with this in the book by Hinze [1].

We will work with $E_1(k_1)$, the energy spectrum of longitudinal velocity fluctuations, and $f(x_1)$, the longitudinal correlation coefficient. Hinze [1] cites Taylor [2] as the source of the cosine-Fourier transform relationship between these two quantities, thus:\begin{equation}U^2 f(x) = \int_0^\infty\, dk_1\,E_1(k_1) \cos(k_1x_1),\end{equation}and \begin{equation}E_1(k_1) =\frac{2}{\pi}\int_0^\infty\, dx_1\, f(x_1) \cos(k_1x_1),\end{equation} where U is the root mean square velocity.

In general, the power laws only apply in the inertial range, which means that we need to restrict the limits of the integrations. However, Hinze obtained a form which allows one to work with the definite limits given above, and reference should be made to page 198 of the first edition of his book [1] for the expression: \begin{equation}U^2\left[1-f(x_1)\right] = C \int_0^\infty\,dk_1\,k_1^{-5/3}\left[1-\cos(k_1x_1)\right],\label{hinze} \end{equation} where C is a universal constant.

The trick he employed to evaluate the right hand side is to make the change of variables:\begin{equation}y= k_1x_1 \quad \mbox{hence} \quad dk_1 =\frac{dy}{x_1}. \end{equation} With this substitution, the right hand side of equation (\ref{hinze}) becomes: \begin{equation}\mbox{RHS of (3)} = C x_1^{2/3}\,\int_0^\infty \,dy [1-\cos y].\end{equation} Integration by parts then leads to: \begin{equation}\int_0^\infty \,dy [1-\cos y]=\frac{3}{2}\int_0^\infty\, dy\, y^{-2/3}\, \sin y =\frac{3}{4}\Gamma(1/3),\end{equation} where Γ is the gamma function. Note that I have omitted any time dependence for sake of simplicity, but of course this is easily added.

[1] J. O. Hinze. Turbulence. McGraw-Hill, New York, 1st edition, 1959. (2nd edition, 1975).
[2] G. I. Taylor. Statistical theory of turbulence. Proc. R. Soc., London, Ser.A, 151:421, 1935.