# From ‘wavenumber murder’ to wavenumber muddle?

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In my post of 20 February 2020, I told of the referee who described my use of Fourier transformation as ‘*the usual wavenumber murder*’. I speculated that the situation had improved over the years due to the use of pseudo-spectral methods in direct numerical simulation, although I was able to quote a more recent example where a referee rejected a paper because he wasn’t comfortable with the idea that structure functions could be evaluated from the corresponding spectra.

However, while it is good to see a growing use of spectral methods, at the same time there are differences between the $x$-space and $k$-space pictures, and this can be confusing. Essentially, the phenomenology of fluid dynamicists has been based on the energy conservation equation in real-space, mostly using structure functions; whereas theorists have worked with the energy balance in wavenumber space as a closure problem for renormalization methods. This separation of activities has gone on over many decades.

For the purpose of this post, I want to look again at the Kolmogorov-Obhukov (1941) theory in $x$-space and $k$-space. Kolmogorov worked in real space and it is convenient to denote his two different derivations of inertial range forms as K41A [1] and K41B [2]. We will concentrate on the second of these, where he derived the well-known `4/5′ law for $S_3(r)$, from the KHE equation. We have quoted this previously and it may be obtained from the book [3] as: \begin{equation}\varepsilon =-\frac{3}{4}\frac{\partial S_2}{\partial t}+\frac{1}{4r^4}\frac{\partial (r^4 S_3)}{\partial r} +\frac{3\nu}{2r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right),\end{equation} and all the symbols have their usual meanings.

In order to solve this equation for $S_3$, Kolmogorov neglected both the time-derivative of $S_2$ and the viscous term, and thus obtained a *de facto* closure. In the case of stationary turbulence the first step is exact but for decaying turbulence it is an approximation for the inertial range which Kolmogorov called *local stationarity*. Later Batchelor referred to this as *equilibrium* [4], which is rather unfortunate as turbulence is the archetypal non-equilibrium problem. In fact Batchelor was carrying on Taylor's idea that the Fourier modes acted as mechanical degrees of freedom and so could be treated by the methods of statistical mechanics. As the classical canon of solved problems in statistical mechanics is limited to thermal equilibrium (normally referred to simply as equilibrium), Batchelor was arguing that Taylor's approach would be valid for the inertial range. In fact it isn't because the modes are strongly coupled and this too is not canonical.

In any case, the neglect of the time-derivative of $S_2$ is a key step and its justification in time-dependent flows poses a problem. More recently, McComb and Fairhurst [5] showed that the neglect of this term cannot be an exact step and also cannot be justified by appeal to large Reynolds numbers or restriction to any particular range of values of $r$. In other words, it is a constant term and its neglect must be justified by either measurement or numerical simulation.

The situation is really quite different in wavenumber space. Here we have the Lin equation which is the Fourier transform of the KHE and takes its simplest form as:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)E(k,t) = T(k,t).$$

where

$$T(k,t)= \int_0^\infty dj\, S(k,j:t),$$

and $S(k,j;t)$ can be expressed in terms of the third-order moment $C_{\alpha\beta\gamma}(\mathbf{j},\mathbf{k-j},\mathbf{-

k};t)$.

One immediate difference is that the KHE is purely local in the variable $r$, whereas the Lin equation is non-local in wavenumber. In fact all Fourier modes are coupled together. We can define the inter-mode energy flux as: \begin{equation}\Pi (\kappa,t) = \int_\kappa^\infty dk\,T(k,t) = -\int_0^\kappa dk \, T(k,t).\end{equation}The criterion for an inertial range of wavenumbers is that the condition $\Pi = \varepsilon$ should hold and this is nowadays referred to as scale invariance. It does not apply in any way to the situation in real space and it has no connection with the concept of *local stationarity* which was renamed *equilibrium* by Batchelor.

Lastly, the interpretation of the time-derivative term in wavenumber space is quite different from that in real space. We may see this by rearranging the Lin equation as: \begin{equation}-T(k,t) = I(k,t) – 2\nu k^2 E(k,t), \quad \mbox{where} \quad I(k,t) = -\frac{\partial E(k,t)}{\partial t}.\label{diff}\end{equation}Evidently for free decay the input term $I(k)$ is positive, and this is actually how Uberoi [6] made the first measurements of the transfer spectrum in grid turbulence. He measured the input term and the viscous term and used equation (\ref{diff}) to evaluate $T(k,t)$.

McComb and Fairhurst [5] pointed out that the constant value of the time derivative term in the limit of infinite Reynolds numbers in $r$-space Fourier transforms to a delta function at the origin in $k$-space. In other words this amounts to a derivation of the form postulated by Edwards [7] (following Batchelor [4]) that the transfer spectrum is given in terms of the Dirac delta function $\delta$ by:\begin{equation}-T(k,t) = \varepsilon \delta(k,t) -\varepsilon \delta(k – \infty,t),\end{equation} in the limit of infinite Reynolds numbers, although the Edwards form was for the stationary case.

This of course is a very extreme situation. The key point to note is that, while the time-derivative of $S_2$ poses a problem for local stationarity in $r$-space, the time-derivative of $E(k,t)$ poses no problem for scale invariance in $k$-space. This is why the $-5/3$ spectrum is so widely observed.


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