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In his 1947 exegesis of Kolmogorov's theory, Batchelor [1] explained the underlying idea of a transfer of energy from large eddies to progressively smaller eddies, until the (local) Reynolds number becomes too small for new eddies to form. He pointed out that the situation had been summarized in a rhyme which he believed was due to L. F. Richardson (no reference given) and which is very well known as:

Big whirls have little whirls, Which feed on their velocity,

And little whirls have lesser whirls, And so on to viscosity!

Incidentally he misquoted 'whirls' as 'whorls', and since then most people seem to have followed suit.

In his discussion Batchelor sometimes followed Kolmogorov, and referred to 'pulsations' while at other times he used the more usual 'eddies'. This variation seems to actually underline the lack of precision of the concept; although, despite this, it is intuitively appealing.

The term 'cascade', with its connotations of a stepwise process, and indeed of localness, is also appealing. According to Eyink and Sreenivasan [2] it was first used by Onsager [3]; but it is his earlier use of the concept of energy transfer in wavenumber space [4] that is truly significant. Obukhov had already obtained the inertial-range spectrum corresponding to Kolmogorov's result for the second-order structure function, but this involved the introduction of an ad hoc eddy viscosity [5]. In [4], Onsager essentially pointed out that the energy flux through modes must be constant in the inertial range. This is the property that is often referred to as *scale invariance.*

The physics of turbulent energy transfer and dissipation can readily be deduced from the equation of motion in wavenumber space; but it is interesting to put oneself in the position of Richardson, looking at (one imagines) snapshots of flow visualizations and creating his mental picture of a cascade of eddies. The equation of motion in real space would have given some limited help perhaps. Evidently the nonlinear term had to be responsible for the creation of new, smaller eddies; and it was known that this term conserved energy. Also, one could deduce that the viscous term would be more significant at the smallest scales. Nevertheless, it was a remarkable achievement to summarise the essence of turbulence in this very persuasive way. So what are its disadvantages?

The first disadvantage, in my view, is that it focuses attention on a single snapshot of the turbulence. Or, in statistical terms, on a single realization. This leads to people drawing conclusions that require a single realization (e.g. the importance of internal intermittency). However, we must always bear in mind that we need average quantities, and to get to them we actually have to take an average. So, if we average our single snapshot of a flow visualization by taking many such snapshots and constructing a form of ensemble average, the result is a blur! For instance, the recent paper by Yoffe and McComb [6] shows how internal intermittency disappears under ensemble averaging.

Paradoxically, my choice for second disadvantage is that I have concluded that the term ‘cascade’ is unhelpful when applied to the inter-mode energy flux. And this, I may say, is despite the fact that I have spent a working lifetime doing just that! In principle, every wavenumber is coupled to every other wavenumber by the nonlinear term. So we can see the attraction of having some sort of cascade or idea of localness. Indeed, in the 1980s/90s there was quite a lot of attention given, using numerical simulations, to the relative importance of different triads of wavenumbers for energy transfer. Now I am not disparaging that work in any way, but it is very complicated and should not distract us from the essential fact: the flux of energy through a wavenumber κ, from *all other wavenumbers less than* κ, is constant for all values of κ in the inertial range. This fact is all the ‘localness’ that we need for the Obukhov-Onsager energy spectrum.

Lastly, it should be understood that the cascade in real space is spread out in space and time. That is, if we distinguish between scale and position by introducing relative and centroid coordinates, thus: $$r= (x-x') \quad \mbox{and}\quad R=(x+x')/2,$$ then in order to observe a cascade through scale r we have to change the position of observation R with time. In contrast, the flux through a mode with wavenumber κ takes place at a single value of R. It is for this reason that the flux in wavenumber space cannot be applied to the cascade in real space.

Still, the term ‘cascade’, in the context of wavenumber space, is so embedded in the general consciousness (including my own!) that there is little possibility of making a change.

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