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Obukhov is regarded as having begun the treatment of the problem in wavenumber space. In [1] he referred to an earlier paper by Kolmogorov for the spectral decomposition of the velocity field in one dimension and pointed out that the three-dimensional case is carried out similarly by multiple Fourier integrals. He employed the Fourier-Stieltjes integral but fortunately this usage did not survive. For many decades the standard Fourier transform has been employed in this field.

[a] Obukhov's paper [1] was published between K41A and K41B, and was described by Batchelor 'as to some extent anticipating the work of Kolmogorov'. He worked with the energy balance in k-space and, influenced by Prandtl's work, introduced an *ad hoc* closure based on an effective viscosity.

[b] The derivation of the '-5/3' law for the energy spectrum seems to have been due to Onsager [2]. He argued that Kolmogorov's similarity principles in x-space would imply an invariant flux (equal to the dissipation) through those wavenumbers where the viscosity could be neglected. Dimensional analysis then led to $E(k) \sim \varepsilon^{2/3}k^{-5/3}$.

[c] As mentioned in the previous post (points [c] and [d]), Batchelor discussed both K41A and K41B in his paper [3], but did not include K41B in his book [4]. Also, in his book [4],

he discussed K41A entirely in wavenumber space. The reasons for this change to a somewhat revisionist approach can only be guessed at, but there may be a clue in his book. On page 29, first paragraph, he says: ‘Fourier analysis of the velocity field provides us with an extremely valuable analytical tool *and one that is well-nigh indispensable for the interpretation of equilibrium or similarity hypotheses*.’ (The emphasis is mine.)

[d] This is a very strong statement, and of course the reference is to Kolmogorov’s theory. There is also the fact that K41B is not easily translated into k-space. Others followed suit, and Hinze [5] actually gave the impression of quoting from K41A but used the word ‘wavenumber’, which does not in fact occur in that work. By the time I began work as a postgraduate student in 1966, the use of spectral methods had become universal in both experiment and theory.

[e] There does not appear to be any k-space *ad hoc* closure of the Lin equation to parallel K41B (i.e. the derivation of the ‘4/5’ law); but, for the specific case of stationary turbulence, I have put forward a treatment which uses the infinite Reynolds number limit to eliminate the energy spectrum, while retaining its effect through the dissipation rate [6]. It is based on the scale invariance of the inertial flux, thus:

$$\Pi(\kappa)=-\int_0^{\kappa}dk\,T(k) = \varepsilon,$$

which of course can be written in terms of the triple-moment of the velocity field. As the velocity field in k-space is complex, we can write it in terms of amplitude and phase. Accordingly,

$$u_{\alpha}(\mathbf{k}) = V(\kappa)\psi_{\alpha}(k'),$$

where $V(\kappa)$ is the root-mean-square velocity, $k'=k/\kappa$ and ψ represents phase effects. The result is:

$$V(\kappa)=B^{-1/3}\varepsilon^{1/3}\kappa^{-10/3},$$

where B is a constant determined by an integral over the triple-moment of the phases of the system.

The Kolmogorov spectral constant is then found to be: $4\pi\, B^{-2/3}$.

[f] Of course a statistical closure, such as the LET theory, is needed to evaluate the expression for B. Nevertheless, it is of interest to note that this theory provides an answer to Kraichnan’s interpretation of Landau’s criticism of K41A [7]. Namely, that the dependence of an average (i.e. the spectrum) on the two-thirds power of an average (i.e. the term involving the dissipation) destroys the linearity of the averaging process. In fact, the minus two-thirds power of the average in the form of $B^{-2/3}$ cancels the dependence associated with the dissipation.

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