Summary of Kolmogorov-Obukhov (1941) theory. Part 1: some preliminaries in x-space and k-space.

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Discussions of the Kolmogorov-Obukhov theory often touch on the question: can the two-thirds *law;* or, alternatively, the *minus five-thirds law,* be derived from the equations of motion (NSE)? And the answer is almost always: ‘no, they can’t’! Yet virtually every aspect of this theory is based on what can be readily deduced from the NSE, and indeed has so been deduced, many years ago. So our preliminary here to the actual summary, is to consider what we know from a consideration of the NSE, in both x-space and k-space. As another preliminary, all the notation is standard and can be found in the two books cited below as references.

We begin with the familiar NSE, consisting of the equation of motion, \begin{equation}\frac{\partial u_{\alpha}}{\partial t} + \frac{\partial (u_{\alpha}u_\beta)}{\partial x_\beta} =- \frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu \nabla^2 u_\alpha,\end{equation}which expresses conservation of momentum and is local, in that it gives the relationship between the various terms at one point in space; and the incompressibility condition \begin{equation}\frac{\partial u_\beta}{\partial x_\beta} = 0.\end{equation} It is well known that taking these two equations together allows us to eliminate the pressure by solving a Poisson-type equation. The result is an expression for the pressure which is an integral over the entire velocity field: see equations (2.3) and (2.9) in [1].

In k-space we may write the Fourier-transformed version of (1) as:

$$\frac{\partial u_\alpha(\mathbf{k},t)}{\partial t} + i k_\beta\int d^3 j u_\alpha(\mathbf{k-j}.t)u_\beta(\mathbf{j},t) = k_\alpha p(\mathbf{k},t) -\nu k^2 u_\alpha (\mathbf{k},t).$$

The derivation can be found in Section 2.4 of [2]. Also, the discrete Fourier-series version (i.e. in finite box) is equation (2.37) in [2].

The crucial point here is that the modes $\mathbf{u}(\mathbf{k},t)$ form a complete set of degrees of freedom and that each mode is coupled to every other mode by the non-linear term. So this is not just a problem in statistical physics, it is an example of the many-body problem.

Note that (1) gives no hint of the cascade, but (3) does. All modes are coupled together and, if there were no viscosity present, this would lead to equipartition, as the conservative non-linear term merely shares out energy among the modes. The viscous term is symmetry-breaking due to the factor k^2 which increases the dissipation as the wavenumber increases. This prevents equipartition and leads to a cascade from low to high wavenumbers. All of this becomes even clearer when we multiply the equation of motion by the velocity and average. We then obtain the energy-balance equations in both x-space and k-space.

We begin in real space with the Karman-Howarth equation (KHE). This can be written in various forms (see Section 3.10.1 in [2]), and here we write in terms of the structure functions for the case of free decay:

$$\varepsilon =-\frac{3}{4}\frac{\partial S_2}{\partial t}+\frac{1}{4r^4}\frac{\partial (r^4 S_3)}{\partial r} +\frac{3\nu}{2r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right).$$

Note that the pressure does not appear, as a correlation of the form $\langle up \rangle$

cannot contribute to an isotropic field, and that strictly the left hand side should be the decay rate ε_D but it is usual to replace this by the dissipation as the two are equal in free decay. Full details of the derivation can be found in Section 3.10 of [2].

For our present purposes, we should emphasise two points. First, this is one equation for two dependent variables and so requires a statistical closure in order to solve for one of the two. In other words, it is an instance of the notorious statistical closure problem. Second, it is local in the variable r and does not couple different scales together. It holds for any value of r but is an energy balance locally at any chosen value of r.

The Lin equation is the Fourier transform of the KHE. It can be derived directly in k-space from the NSE (see Section 3.2.1 in [2]):\begin{equation} \left(\frac{\partial}{\partial t} + 2\nu k^2\right)E(k,t) = T(k,t).\end{equation} Here $T(k,t)$ is called the *transfer spectrum*, and can be written as: \begin{equation}T(k,t)= \int_0^\infty dj\, S(k,j:t),\end{equation}where $S(k,j;t)$ is the *transfer spectral density* and can be expressed in terms of the third-order moment $C_{\alpha\beta\gamma}(\mathbf{j},\mathbf{k-j},\mathbf{-k};t)$.

Unlike the KHE, which is purely local in its independent variable, the Lin equation is non-local in wavenumber. We can define its associated inter-mode energy flux as:\begin{equation}\Pi (\kappa,t) = \int_\kappa^\infty dk\,T(k,t) = -\int_0^\kappa dk \, T(k,t).\end{equation}

We have now laid a basis for a summary of the Kolmogorov-Obhukov theory and one point should have emerged clearly: the energy cascade is well defined in wavenumber space. It is not defined at all in the context of energy conservation in real space. It can only exist as an intuitive phenomenon which is extended in space and time.

[1] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.
[2] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.