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To which you might be tempted to reply: 'Who ever thought it was?' Well, I don't know for sure, but I've developed a suspicion that such a misconception may underpin the belief that it is necessary to specify that turbulence is homogeneous as well as isotropic. When I began my career it was widely understood that specifying isotropy was sufficient, as it was generally realised that homogeneity was a necessary condition for isotropy. A statement to this effect could (and can) be found on page 3 of Batchelor's famous monograph on the subject [1].

I have posted previously on this topic (my second post, actually, on 12 February 2020) and conceded that the acronym HIT, standing of course for '*homogeneous, isotropic turbulence*', has its attractions. For a start, it's the shortest possible way of telling people that you are concerned with isotropic turbulence. I've used it myself and will probably continue to do so. So I don't see anything wrong with using it, as such. The problem arises, I think, when some people think that you *must* use it. In other words, such people apparently believe that there is an inhomogeneous form of isotropic turbulence.

When you think about it that is really quite worrying. I'm not particularly happy about someone, whose understanding is so limited, refereeing one of my papers. Although, to be honest, that could well explain some of the more bizarre referees' reports over the years! Anyway, let's examine the idea that there may be some confusion between isotropy and spherical symmetry.

Isotropy just means that a property is independent of orientation. Spherical symmetry sounds quite similar and is probably the more frequently encountered concept for most of us (at least during our formal education). Essentially it means that, relative to some fixed point, a field only varies with distance from the point but not with angle. A familiar example would be a point electric charge in free space. So we might be tempted to visualise isotropy as a form of spherical symmetry, the common element being the independence of orientation.

The problem with doing this, is that the property of isotropy of a medium must apply to any point within it. Whereas, spherical symmetry depends on the existence of a special point which may be taken as the origin of coordinates. But the existence of such a special point would violate spatial homogeneity. So for isotropy to be true, we must have spatial uniformity or homogeneity. I think that one can infer this mathematically from the fact that the only isotropic tensors are (subject to a scalar multiplier) the Kronecker delta δ_{ij} and the Levi-Civita density ϵ_{ijk} . So any isotropic tensor must have components that are independent of the coordinates of the system.

For this point applied to the cosmos, i.e. homogeneity is a necessary (but not sufficient) condition for isotropy, see Figure 2 on page 24 of [2]. It seems to be easier to visualise these matters in terms of the night sky which is a fairly (if, illusory) static-looking entity. But when we add in a continuum structure and random variations on many length scales, it can be more difficult. We will come back to this particular problem in my next post.

[1] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 2nd edition, 1971.

[2] Steven Weinberg. The first three minutes: a modern view of the origin of the universe. Basic Books, NY, 1993.