The Kolmogorov (1962) theory: a critical review Part 1

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As is well known, Kolmogorov interpreted Landau's criticism as referring to the small-scale intermittency of the instantaneous dissipation rate. His response was to adopt Obukhov's proposal to introduce a new dissipation rate which had been averaged over a sphere of radius r, and which may be denoted by ε_r. This procedure runs into an immediate fundamental objection.

In K41A, (or its wavenumber-space equivalent) the relevant inertial-range quantity for the dimensional analysis is the local (in wavenumber) energy transfer. This is of course equal to the mean dissipation rate by the global conservation of energy (It is a potent source of confusion that these theories are almost always discussed in terms of the dissipation ε, when the proper inertial-range quantity is the nonlinear transfer of energy Π. The inertial range is defined by the condition $\Pi_{max} = \varepsilon$). However, as pointed out by Kraichnan [1] there is no such simple relationship between locally-averaged energy transfer and locally-averaged dissipation.

Although Kolmogorov presented his 1962 theory as `A refinement of previous hypotheses …', it is now generally understood that this is incorrect. In fact it is a radical change of approach. The 1941 theory amounted to a *general assumption* that a cascade of many steps would lead to scales where the mean properties of turbulence were independent of the conditions of formation (i.e. of, essentially, the physical size of the system). Whereas, in 1962, the assumption was, in effect, that the mean properties of turbulence *did* depend on the physical size of the system. We will return to this point presently, but for the moment we concentrate on the preliminary steps.

The 1941 theory relied on a general assumption with an underlying physical plausibility. In contrast, the 1962 theory involved an arbitrary and specific assumption. This was to the effect that the logarithm of $\varepsilon(\mathbf{x},t)$ has a normal distribution for large L/r where L is referred to as an external scale and is related to the physical size of the system. We describe this as `arbitrary' because no physical justification is offered; but in any case it is certainly specific. Then, arguments were developed that led to a modified expression for the second-order structure function, thus: \begin{equation}S_2(r)=C(\mathbf{x},t)\varepsilon^{2/3}r^{2/3}(L/r)^{-\mu}, \label{62S2}\end{equation} where $C(\mathbf{x},t)$ depends on the macrostructure of the flow.

In addition, Kolmogorov pointed out that *`the theorem of constancy of skewness …derived (sic) in Kolmogorov (1941b)'* is replaced by \begin{equation} S(r) = S_0(L/r)^{3\mu/2},\end{equation} where S_0 also depends on the macrostructrure.

Equation (\ref{62S2}) is rather clumsy in structure, in the way the prefactor C depends on x. This is because of course we have $r=x-x'$, so clearly $C(\mathbf{x},t)$ also depends on r. A better way of tackling this would be to introduce centroid and relative coordinates, $\mathbf{R}$ and $\mathbf{r}$, such that \begin{equation}\mathbf{R} = (\mathbf{x}+\mathbf{x'})/2; \qquad \mbox{and} \qquad \mathbf{r}= (\mathbf{x}-\mathbf{x'}).\end{equation} Then we can re-write the prefactor as $C(\mathbf{R}, r; t)$, where the dependence on the macrostructure is represented by the centroid variable, while the dependence on the relative variable holds out the possibility that the prefactor becomes constant for sufficiently small values of r.

Of course, if we restrict our attention to homogeneous fields, then there can be no dependence of mean quantities on the centroid variable. Accordingly, one should make the replacement: \begin{equation}C(\mathbf{R}, r; t)=C(r; t),\end{equation} and the additional restriction to stationarity would eliminate the dependence on time. In fact Kraichnan [1] went further and replaced the pre-factor with the constant C: see his equation (1.9).

For sake of completeness, another point worth mentioning at this stage is that the derivation of the `4/5′ law is completely unaffected by the `refinements' of K62. This is really rather obvious. The Karman-Howarth equation involves only ensemble-averaged quantities and the derivation of the `4/5′ law requires only the vanishing of the viscous term. This fact was noted by Kolmogorov [2].

[1] R. H. Kraichnan. On Kolmogorov's inertial-range theories. J. Fluid Mech., 62:305, 1974.
[2] A. N. Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech., 13:82-85, 1962.

The Landau criticism of K41 and problems with averages

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The idea that K41 had some problem with the way that averages were taken has its origins in the famous footnote on page 126 of the book by Landau and Lifshitz [1]. This footnote is notoriously difficult to understand; not least because it is meaningless unless its discussion of the `dissipation rate $\varepsilon$' refers to the instantaneous dissipation rate. Yet $\varepsilon$ is clearly defined in the text above (see the equation immediately before their (33.8)) as being the mean dissipation rate. Nevertheless, the footnote ends with the sentence `The result of the averaging therefore cannot be universal'. As their preceding discussion in the footnote makes clear, this lack of universality refers to 'different flows': presumably wakes, jets, duct flows, and so on.

We can attempt a degree of deconstruction as follows. We will use our own notation, and to this end we introduce the instantaneous structure function $\hat{S}_2(r,t)$, such that $\langle \hat{S}_2(r,t) \rangle =S_2(r)$. Landau and Lifshitz consider the possibility that $S_2(r)$ could be a universal function in any turbulent flow, for sufficiently small values of r (i.e. the Kolmogorov theory). They then reject this possibility, beginning with the statement:

`The instantaneous value of $\hat{S}(r,t)$ might in principle be expressed as a universal function of the energy dissipation ε at the instant considered.'

Now this is rather an odd statement. Ignoring the fact that the dissipation is not the relevant quantity for inertial-range behaviour, it is really quite meaningless to discuss the universality of a random variable in terms of its relation to a mean variable (i.e. the dissipation). A discussion of universality requires mean quantities. Otherwise it is impossible to test the statement. The authors have possibly relied on the qualification `at the instant considered'. But how would one establish which instant that was for various different flows?

They then go on:

`When we average these expressions, however, an important part will be played by the law of variation of ε over times of the order of the periods of the large eddies (of size $\sim L$), and this law is different for different flows.'

This seems a rather dogmatic statement but it is clearly wrong for the the broad (and important) class of stationary flows. In such flows, ε does not vary with time.

The authors conclude (as we pointed out above) that: `The result of the averaging therefore cannot be universal.' One has to make allowance for possible uncertainties arising in translation, but nevertheless, the latter part of their argument only makes any sort of sense if the dissipation rate is also instantaneous. Such an assumption appears to have been made by Kraichnan [2], who provided an interpretation which does not actually depend on the nature of the averaging process.

In fact Kraichnan worked with the energy spectrum, rather than the structure function, and interpreted Landau's criticism of K41 as applying to \begin{equation}E(k) = \alpha\varepsilon^{2/3}k^{-5/3}.\label{6-K41}\end{equation} His interpretation of Landau was that the prefactor α may not be a universal constant because the left-hand side of equation (\ref{6-K41}) is an average, while the right-hand side is the 2/3 power of an average.

Any average involves the taking of a limit. Suppose we consider a time average, then we have \begin{equation} E(k) = \lim_{T\rightarrow\infty}\frac{1}{T}\int^{T}_{0}\widehat{E}(k, t)dt, \end{equation} where as usual the `hat' denotes an instantaneous value. Clearly the statement \begin{equation}E(k) = \mbox{a constant};\end{equation}or equally the statement, \begin{equation}E(k) = f\equiv\langle\hat{f}\rangle, \end{equation} for some suitable $f$, presents no problem. It is the `2/3´ power on the right-hand side of equation (\ref{6-K41}) which means that we are apparently equating the operation of taking a limit to the 2/3 power of taking a limit.

However, it has recently been shown [3] that this issue is resolved by noting that the pre-factor α itself involves an average over the phases of the system. It turns out that α depends on an ensemble average to the $-2/3$ power and this cancels the dependence on the $2/3$ power on the right hand side of (\ref{6-K41}).

[1] L. D. Landau and E. M. Lifshitz. Fluid Mechanics. Pergamon Press, London, English edition, 1959.
[2] R. H. Kraichnan. On Kolmogorov's inertial-range theories. J. Fluid Mech., 62:305, 1974.
[3] David McComb. Scale-invariance and the inertial-range spectrum in three-dimensional stationary, isotropic turbulence. J. Phys. A: Math. Theor.,42:125501, 2009.

The Kolmogorov-Obukhov Spectrum.

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To lay a foundation for the present piece, we will first consider the joint Kolmogorov-Obukhov picture in more detail. For completeness, we should begin by mentioning that Kolmogorov also used the Karman-Howarth equation, which is the energy balance equation connecting the second- and third-order structure functions, to derive the so-called \`$4/5$' law for the third-order structure function.This procedure amounts to a *de facto* closure, as the time-derivative is neglected (an exact step in our present case, as we are restricting our attention to stationary turbulence) and the term involving the viscosity vanishes in the limit of infinite Reynolds number. This is often referred to as \`the only exact result in turbulence theory'; but increasingly it is being referred to, perhaps more correctly, as \`the only *asymptotically* exact result in turbulence'.

As part of this work, he also assumed that the skewness was constant; and this provided a relationship between the second- and third-order structure functions which recovered the `$2/3$' law. It is interesting to note that Lundgren used the method of matched asymptotic expansions to obtain both the `$4/5$' and `$2/3$' laws, without having to make any assumption about the skewness. This work also offered a way of estimating the extent of the inertial range in real space.

However, the Karman-Howarth equation is local in the independent variables and therefore does not describe an energy cascade. In contrast, the Lin equation (which is just its Fourier transform) shows that all the degrees of freedom in turbulence are coupled together. It takes the form, for the energy spectrum $E(k, t)$, in the presence of an input spectrum $W(k)$: \begin{equation}\frac{\partial E(k,t)}{\partial t} = W(k)+ T(k,t)- 2\nu_{0}k^{2}E(k, t),\label{lin}\end{equation} where ν_{0} is the kinematic viscosity and the transfer spectrum $T(k,t)$ is given by\begin{eqnarray}T(k,t) & = & 2\pi k^{2}\int d^{3}j\int d^{3}l\,\delta(\mathbf{k}-\mathbf{j}-\mathbf{l})M_{\alpha\beta\gamma}(\mathbf{k})\nonumber \\ & \times & \left\{C_{\beta\gamma\alpha}(\mathbf{j},\mathbf{l},\mathbf{-k};t)-C_{\beta\gamma\alpha}(\mathbf{-j},\mathbf{-l},\mathbf{k};t)\right\},\end{eqnarray}with \begin{equation} M_{\alpha\beta\gamma}(\mathbf{k})=-\frac{i}{2}\left[k_{\beta}P_{\alpha\gamma}(\mathbf{k})+k_{\gamma}P_{\alpha\beta}(\mathbf{k})\right],\label{M}\end{equation} and the projector $P_{\alpha\beta}(\mathbf{k})$ is \begin{equation}P_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-\frac{k_{\alpha}k_{\beta}}{|\mathbf{k}|^{2}}, \end{equation}where $\delta_{\alpha\beta}$ is the Kronecker delta, and the third-order moment $C_{\beta\gamma\alpha}$ here takes the specific form: \begin{equation} C_{\beta\gamma\alpha}(\mathbf{j},\mathbf{l},\mathbf{-k};t)=\langle u_{\beta}(\mathbf{j},t)u_{\gamma}(\mathbf{l},t)u_{\alpha}(\mathbf{-k},t) \rangle.\end{equation}

At this stage we also define the flux of energy $\Pi(\kappa,t)$ due to inertial transfer through the mode with wavenumber $k=\kappa$. This is given by: \begin{equation}\Pi(\kappa,t) = \int_{\kappa}^{\infty}\,dk\,T(k,t).\end{equation}
Further discussion and details may be found in Section 4.2 of the book [1].
We now have a rather simple picture. In formulating our problem, the shape of the input spectrum should be chosen to be peaked near the origin, such that higher wavenumbers are driven by inertial transfer, with energy being dissipated locally by the viscosity. Then we can define the rate at which stirring forces do work on the system by: \begin{equation} \int_0^\infty \, W(k)\, dk = \varepsilon_W. \end{equation}

Obukhov’s idea of the constant inertial flux can be expressed as follows. As the Reynolds number is increased, the transfer rate, as given by equation (6), will also increase and must reach a maximum value, which in turn must be equal to the viscous dissipation. Thus we introduce the symbol ε_T for the maximum inertial flux as: \begin{equation}\varepsilon_T = \Pi_{\mbox{max}},\end{equation} and for stationary turbulence at sufficiently high Reynolds number, we have the limiting condition: \begin{equation}\varepsilon = \varepsilon_T = \varepsilon_W.\end{equation}

Thus the loose idea of a local cascade involving eddies in real space is replaced by the precisely formulated concept of *scale invariance* of the inertial flux in wavenumber space. As is well known, this picture leads directly to the $-5/3$ energy spectrum in the limit of large Reynolds numbers.

[1] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures.

Oxford University Press, 2014.

Why do we call it 'The Kolmogorov Spectrum'?

Why do we call it 'The Kolmogorov Spectrum'?
The Kolmogorov $-5/3$ spectrum continues to be the subject of contentious debate. Despite its great utility in applications and its overwhelming confirmation by experiments, it is still plagued by the idea that it is subject to intermittency corrections. From a fundamental view this is difficult to understand because Kolmogorov's theory (K41a) was expressed in terms of the mean dissipation, which can hardly be affected by intermittency. Another problem is that Kolmogorov actually derived the $2/3$ law for the structure function. Of course one can derive the spectrum from this result by Fourier transformation; but this is not a completely trivial process and we will discuss it in a future post.

The trouble seems to be that Kolmogorov's theory, despite its great pioneering importance, was an incomplete and inconsistent theory. It was formulated in real space; where, although the energy transfer process can be loosely visualised from Richardson's idea of a cascade, the concept of such a cascade is not mathematically well defined. Also, having introduced the inertial range of scales, where the viscosity may be neglected, he characterised this range by the viscous dissipation rate, which is not only inconsistent but incorrect. An additional complication, which undoubtedly plays a part, is that his theory was applied to turbulence in general. The basic idea was that the largest scales would be affected by the nature of the flow, but a stepwise cascade would result in smaller eddies being universal in some sense. That is, they would have much the same statistical properties, despite the different conditions of formation. In order to avoid uncertainties that can arise from this rather general idea, we will restrict our attention to stationary, isotropic turbulence here.

To make a more physical picture we have to follow Obukhov and work in k space with the Fourier transform $\mathbf{u}(\mathbf{k},t)$ of the velocity field $\mathbf{u}(\mathbf{x},t)$. This was introduced by Taylor in order to allow the problem of isotropic turbulence to be formulated as one of statistical mechanics, with the Fourier components acting as the degrees of freedom. In this way, Obukhov identified the conservative, inertial flux of energy through the modes as being the key quantity determining the energy spectrum in the inertial range. It follows that, with the input and dissipation being negligible, the flux must be constant (i.e. independent of wavenumber) in the inertial range, with the extent of the inertial range increasing as the Reynolds number was increased, and this was later recognized by Osager in (1945). Later still, this property became widely known and for many years has been referred to by theoretical physicists as scale invariance. It should be emphasised that the inertial flux is an average quantitiy, as indeed is the energy spectrum, and any intermittency effects present, which are characteristics of the instantaneous velocity field, will inevitably be averaged out. Of course, in stationary flows the inertial transfer rate is the same as the dissipation rate, but in non-stationary flows it is not.

This is not intended to minimise the importance of Kolmogorov’s pioneering work. It is merely that we would argue that one also needs to consider Obukhov’s theory (also, in 1941), with possibly also a later contribution from Onsager (in 1945), in order to have a complete theoretical picture. In effect this seems to have been the view of the turbulence community from the late 1940s onwards. Discussion of turbulent energy transfer and dissipation in isotropic turbulence was almost entirely in terms of the spectral picture. It was not until the extensive measurements of higher-order structure functions by Anselmet et al. (in 1984) that the real-space picture became of interest, along with the concept of anomalous exponents.

I would argue that we should go back to the term ‘Kolmogorov-Obukhov spectrum’, as indeed was quite often done in earlier years. We will develop this idea in the next post. All source references for this piece will be found in the book [1].

[1] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.