The Kolmogorov (1962) theory: a critical review Part 1

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As is well known, Kolmogorov interpreted Landau's criticism as referring to the small-scale intermittency of the instantaneous dissipation rate. His response was to adopt Obukhov's proposal to introduce a new dissipation rate which had been averaged over a sphere of radius r, and which may be denoted by ε_r. This procedure runs into an immediate fundamental objection.

In K41A, (or its wavenumber-space equivalent) the relevant inertial-range quantity for the dimensional analysis is the local (in wavenumber) energy transfer. This is of course equal to the mean dissipation rate by the global conservation of energy (It is a potent source of confusion that these theories are almost always discussed in terms of the dissipation ε, when the proper inertial-range quantity is the nonlinear transfer of energy Π. The inertial range is defined by the condition $\Pi_{max} = \varepsilon$). However, as pointed out by Kraichnan [1] there is no such simple relationship between locally-averaged energy transfer and locally-averaged dissipation.

Although Kolmogorov presented his 1962 theory as `A refinement of previous hypotheses …', it is now generally understood that this is incorrect. In fact it is a radical change of approach. The 1941 theory amounted to a *general assumption* that a cascade of many steps would lead to scales where the mean properties of turbulence were independent of the conditions of formation (i.e. of, essentially, the physical size of the system). Whereas, in 1962, the assumption was, in effect, that the mean properties of turbulence *did* depend on the physical size of the system. We will return to this point presently, but for the moment we concentrate on the preliminary steps.

The 1941 theory relied on a general assumption with an underlying physical plausibility. In contrast, the 1962 theory involved an arbitrary and specific assumption. This was to the effect that the logarithm of $\varepsilon(\mathbf{x},t)$ has a normal distribution for large L/r where L is referred to as an external scale and is related to the physical size of the system. We describe this as `arbitrary' because no physical justification is offered; but in any case it is certainly specific. Then, arguments were developed that led to a modified expression for the second-order structure function, thus:
\begin{equation}S_2(r)=C(\mathbf{x},t)\varepsilon^{2/3}r^{2/3}(L/r)^{-\mu}, \label{62S2}\end{equation} where $C(\mathbf{x},t)$ depends on the macrostructure of the flow.

In addition, Kolmogorov pointed out that *`the theorem of constancy of skewness …derived (sic) in Kolmogorov (1941b)'* is replaced by \begin{equation} S(r) = S_0(L/r)^{3\mu/2},\end{equation} where S_0 also depends on the macrostructrure.

Equation (\ref{62S2}) is rather clumsy in structure, in the way the prefactor C depends on x. This is because of course we have $r=x-x'$, so clearly $C(\mathbf{x},t)$ also depends on r. A better way of tackling this would be to introduce centroid and relative coordinates, $\mathbf{R}$ and $\mathbf{r}$, such that \begin{equation}\mathbf{R} = (\mathbf{x}+\mathbf{x'})/2; \qquad \mbox{and} \qquad \mathbf{r}= (\mathbf{x}-\mathbf{x'}).\end{equation} Then we can re-write the prefactor as $C(\mathbf{R}, r; t)$, where the dependence on the macrostructure is represented by the centroid variable, while the dependence on the relative variable holds out the possibility that the prefactor becomes constant for sufficiently small values of r.

Of course, if we restrict our attention to homogeneous fields, then there can be no dependence of mean quantities on the centroid variable. Accordingly, one should make the replacement: \begin{equation}C(\mathbf{R}, r; t)=C(r; t),\end{equation} and the additional restriction to stationarity would eliminate the dependence on time. In fact Kraichnan [1] went further and replaced the pre-factor with the constant C: see his equation (1.9).

For sake of completeness, another point worth mentioning at this stage is that the derivation of the `4/5′ law is completely unaffected by the `refinements’ of K62. This is really rather obvious. The Karman-Howarth equation involves only ensemble-averaged quantities and the derivation of the `4/5′ law requires only the vanishing of the viscous term. This fact was noted by Kolmogorov [2].

[1] R. H. Kraichnan. On Kolmogorov’s inertial-range theories. J. Fluid Mech., 62:305, 1974.
[2] A. N. Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech., 13:82-85, 1962.