The Landau criticism of K41 and problems with averages

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The idea that K41 had some problem with the way that averages were taken has its origins in the famous footnote on page 126 of the book by Landau and Lifshitz [1]. This footnote is notoriously difficult to understand; not least because it is meaningless unless its discussion of the `dissipation rate $\varepsilon$' refers to the instantaneous dissipation rate. Yet $\varepsilon$ is clearly defined in the text above (see the equation immediately before their (33.8)) as being the mean dissipation rate. Nevertheless, the footnote ends with the sentence `The result of the averaging therefore cannot be universal'. As their preceding discussion in the footnote makes clear, this lack of universality refers to 'different flows': presumably wakes, jets, duct flows, and so on.

We can attempt a degree of deconstruction as follows. We will use our own notation, and to this end we introduce the instantaneous structure function $\hat{S}_2(r,t)$, such that $\langle \hat{S}_2(r,t) \rangle =S_2(r)$. Landau and Lifshitz consider the possibility that $S_2(r)$ could be a universal function in any turbulent flow, for sufficiently small values of r (i.e. the Kolmogorov theory). They then reject this possibility, beginning with the statement:

`The instantaneous value of $\hat{S}(r,t)$ might in principle be expressed as a universal function of the energy dissipation ε at the instant considered.'

Now this is rather an odd statement. Ignoring the fact that the dissipation is not the relevant quantity for inertial-range behaviour, it is really quite meaningless to discuss the universality of a random variable in terms of its relation to a mean variable (i.e. the dissipation). A discussion of universality requires mean quantities. Otherwise it is impossible to test the statement. The authors have possibly relied on the qualification `at the instant considered’. But how would one establish which instant that was for various different flows?

They then go on:

`When we average these expressions, however, an important part will be played by the law of variation of ε over times of the order of the periods of the large eddies (of size $\sim L$), and this law is different for different flows.’

This seems a rather dogmatic statement but it is clearly wrong for the the broad (and important) class of stationary flows. In such flows, ε does not vary with time.

The authors conclude (as we pointed out above) that: `The result of the averaging therefore cannot be universal.’ One has to make allowance for possible uncertainties arising in translation, but nevertheless, the latter part of their argument only makes any sort of sense if the dissipation rate is also instantaneous. Such an assumption appears to have been made by Kraichnan [2], who provided an interpretation which does not actually depend on the nature of the averaging process.

In fact Kraichnan worked with the energy spectrum, rather than the structure function, and interpreted Landau’s criticism of K41 as applying to \begin{equation}E(k) = \alpha\varepsilon^{2/3}k^{-5/3}.\label{6-K41}\end{equation} His interpretation of Landau was that the prefactor α may not be a universal constant because the left-hand side of equation (\ref{6-K41}) is an average, while the right-hand side is the 2/3 power of an average.

Any average involves the taking of a limit. Suppose we consider a time average, then we have \begin{equation} E(k) =

\lim_{T\rightarrow\infty}\frac{1}{T}\int^{T}_{0}\widehat{E}(k, t)dt, \end{equation} where as usual the `hat’ denotes an instantaneous value. Clearly the statement \begin{equation}E(k) = \mbox{a constant};\end{equation}or equally the statement, \begin{equation}E(k) = f\equiv\langle\hat{f}\rangle, \end{equation} for some suitable $f$, presents no problem. It is the `2/3′ power on the right-hand side of equation (\ref{6-K41}) which means that we are apparently equating the operation of taking a limit to the 2/3 power of taking a limit.

However, it has recently been shown [3] that this issue is resolved by noting that the pre-factor α itself involves an average over the phases of the system. It turns out that α depends on an ensemble average to the $-2/3$ power and this cancels the dependence on the $2/3$ power on the right hand side of (\ref{6-K41}).

[1] L. D. Landau and E. M. Lifshitz. Fluid Mechanics. Pergamon Press, London, English edition, 1959.
[2] R. H. Kraichnan. On Kolmogorov’s inertial-range theories. J. Fluid Mech., 62:305, 1974.
[3] David McComb. Scale-invariance and the inertial-range spectrum in three-dimensional stationary, isotropic turbulence. J. Phys. A: Math. Theor.,42:125501, 2009.