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The Kolmogorov $-5/3$ spectrum continues to be the subject of contentious debate. Despite its great utility in applications and its overwhelming confirmation by experiments, it is still plagued by the idea that it is subject to intermittency corrections. From a fundamental view this is difficult to understand because Kolmogorov's theory (K41a) was expressed in terms of the mean dissipation, which can hardly be affected by intermittency. Another problem is that Kolmogorov actually derived the $2/3$ law for the structure function. Of course one can derive the spectrum from this result by Fourier transformation; but this is not a completely trivial process and we will discuss it in a future post.

The trouble seems to be that Kolmogorov's theory, despite its great pioneering importance, was an incomplete and inconsistent theory. It was formulated in real space; where, although the energy transfer process can be loosely visualised from Richardson's idea of a cascade, the concept of such a cascade is not mathematically well defined. Also, having introduced the inertial range of scales, where the viscosity may be neglected, he characterised this range by the viscous dissipation rate, which is not only inconsistent but incorrect. An additional complication, which undoubtedly plays a part, is that his theory was applied to turbulence in general. The basic idea was that the largest scales would be affected by the nature of the flow, but a stepwise cascade would result in smaller eddies being universal in some sense. That is, they would have much the same statistical properties, despite the different conditions of formation. In order to avoid uncertainties that can arise from this rather general idea, we will restrict our attention to stationary, isotropic turbulence here.

To make a more physical picture we have to follow Obukhov and work in k space with the Fourier transform $\mathbf{u}(\mathbf{k},t)$ of the velocity field $\mathbf{u}(\mathbf{x},t)$. This was introduced by Taylor in order to allow the problem of isotropic turbulence to be formulated as one of statistical mechanics, with the Fourier components acting as the degrees of freedom. In this way, Obukhov identified the conservative, inertial flux of energy through the modes as being the key quantity determining the energy spectrum in the inertial range. It follows that, with the input and dissipation being negligible, the flux must be constant (i.e. independent of wavenumber) in the inertial range, with the extent of the inertial range increasing as the Reynolds number was increased, and this was later recognized by Osager in (1945). Later still, this property became widely known and for many years has been referred to by theoretical physicists as scale invariance. It should be emphasised that the inertial flux is an average quantitiy, as indeed is the energy spectrum, and any intermittency effects present, which are characteristics of the instantaneous velocity field, will inevitably be averaged out. Of course, in stationary flows the inertial transfer rate is the same as the dissipation rate, but in non-stationary flows it is not.

This is not intended to minimise the importance of Kolmogorov’s pioneering work. It is merely that we would argue that one also needs to consider Obukhov’s theory (also, in 1941), with possibly also a later contribution from Onsager (in 1945), in order to have a complete theoretical picture. In effect this seems to have been the view of the turbulence community from the late 1940s onwards. Discussion of turbulent energy transfer and dissipation in isotropic turbulence was almost entirely in terms of the spectral picture. It was not until the extensive measurements of higher-order structure functions by Anselmet et al. (in 1984) that the real-space picture became of interest, along with the concept of anomalous exponents.

I would argue that we should go back to the term ‘Kolmogorov-Obukhov spectrum’, as indeed was quite often done in earlier years. We will develop this idea in the next post. All source references for this piece will be found in the book [1].

[1] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.