The different roles of the Gaussian pdf in Renormalized Perturbation Theory (RPT) and Self-Consistent Field (SCF) theory.

The different roles of the Gaussian pdf in Renormalized Perturbation Theory (RPT) and Self-Consistent Field (SCF) theory.

In last week's blog, I discussed the Kraichnan and Wyld approaches to the turbulence closure problem. These field-theoretic approaches are examples of RPTs, while the pioneering theory of Edwards [1] is a self-consistent field theory. An interesting difference between them is the different ways in which they make use of a Gaussian (or normal) base distribution. Any theory is going to begin with a Gaussian distribution, because it is tractable. We know how to express all its moments in terms of the second-order moment. Of course, we also know that it predicts that odd order moments are zero, so some trick must be employed to get it to tell us anything about turbulence.

As we did last week, we begin with the Fourier-transformed solenoidal Navier-Stokes equation (NSE) written in an extremely compressed notation as: \begin{equation} \mathcal{L}_{0,k}u_k = \lambda M_{0,k}u_ju_{k-j},\end{equation} where the linear operator $\mathcal{L}_{0,k} = \partial /\partial t + \nu_0 k^2$, ν_0 is the kinematic viscosity of the fluid, $M_{0,k}$ is the inertial transfer operator which contains the eliminated pressure term, and λ is a book-keeping parameter which is used to keep track of terms during an iterative solution.

Now let us consider the closure problem. We multiply equation (1) through by u_{-k} and average, to obtain: \begin{equation} \mathcal{L}_{0,k}\langle u_k u_{-k}\rangle= \lambda M_{0,k}\langle u_ju_{k-j}u_{-k}\rangle,\end{equation} where the angle brackets denote an average. Evidentally, if we evaluate the averages here with a Gaussian pdf, the triple moment vanishes (trivially, by symmetry)

Then we set up a perturbation-type approach by expanding the velocity field in powers of λ as: \begin{equation} u_k = u^{(0)}_k + \lambda u^{(1)}_k + \lambda^2 u^{(2)}_k + \lambda^3 u^{(3)}_k + \dots, \end{equation} where $u^{(0)}_k$ is a velocity field with a Gaussian distribution. The general procedure has two steps. First, substitute the expansion (3) into the right hand side of equation (1) and calculate the coefficients iteratively in terms of the $u^{(0)}_k$. Secondly, substitute the explicit form of the expansion, now entirely expressed in terms of the $u^{(0)}$ into the right hand side of equation (2), and evaluate the averages to all orders, using the rules for a Gaussian distribution. If we denote the inverse of the linear operator by $\mathcal{L}^{-1}_{0,k} \equiv R_{0,k}$, and the Gaussian zero-order covariance by $\langle u_k u_{-k}\rangle=C_{0,k}$, then the triple moment on the right hand of equation (2) can be written to all orders in products and convolutions of $R_{0,k}$ and $C_{0,k}$.

Kraichnan introduced renormalization in this problem by making the replacements: \[R_{0,k}\rightarrow R_{k} \quad \mbox{and} \quad C_{0,k} \rightarrow C_k,\] to all orders in the perturbation expansion of the triple-moment in (2). This step involves partial summations of the perturbation expansion in different classes of terms.
At this point it is worth noting that what happens here is rather like in a direct-numerical simulation of the NSE. There we begin with a Gaussian initial field. As time goes on, the nonlinear term induces couplings between modes and the system

moves to a field which is representative of Navier-Stokes turbulence. Of course the initial distribution is constrained in this case to give the total energy that we require in the simulation. Note that the zero-order field in perturbation theory is in principle present at all times and is not constrained in this way.

In contrast, what Edwards introduced was a perturbation expansion of the probability distribution function of the velocity field, not of the velocity field itself. For this reason, he did not work directly with the NSE but instead used it to derive a Liouville equation for the probability distribution $P[u,t]$. It should be noted that the Liouville equation, although containing the nonlinearity of the velocity field, is nevertheless a linear equation for the pdf. Edwards then expanded $P[u,t]$, the exact pdf, as follows:

$$P[u,t] = P^{0}[u] + \epsilon P^{1}[u,t] + \epsilon^2 P^{2}[u,t] + \mathcal{O}(\epsilon^3),$$

where $P^{0}[u]$ is a Gaussian distribution. The significant step here is to demand that the zero-order pdf gives the same result for the second-order moment as the exact pdf. That is,

$$\int \, P^{(0)}[u] \, u_ku_{-k} \mathcal{D}u = \int \, P[u,t] \, u_ku_{-k} \mathcal{D}u \equiv C_k.$$

This is in fact the basis of the self-consistency requirement in the theory. For further details the interested reader should consult either of the books referenced below as [1] and [2]. The Edwards method [3] does not rely on partially summing infinite perturbation series, nor is it like the functional formalisms which are equivalent to such summation procedures. Instead it relies on the fact that the measured pdf in turbulence is not very different from a Gaussian. In this respect, it is encouraging that it gives similar results to the RPTs. This resemblance is heightened in the recent derivation of the LET theory as a two-time SCF [4], thus extending the Edwards method.

[1] D. C. Leslie. Developments in the theory of turbulence. Clarendon Press Oxford, 1973.
[2] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.
[3] S. F. Edwards. The statistical dynamics of homogeneous turbulence. J. Fluid Mech., 18:239, 1964.
[4] W. D. McComb and S. R. Yoffe. A formal derivation of the local energy transfer (LET) theory of homogeneous turbulence. J. Phys. A: Math. Theor., 50:375501, 2017.