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When I began my PhD in 1966, I found Wyld’s paper [1] to be one of the easiest to understand. However, one feature of the formalism struck me as odd or incorrect, so I didn’t spend any more time on it. But I had found it very useful in helping me to understand how a theory like Kraichnan’s DIA could work. In short, I thought that it had pedagogic value. Some years later, when I wrote up my first attempt to derive a two-time version of the LET theory [2], I made use of a variant of Wyld’s formalism, albeit with his procedural error corrected. I was surprised by the hostility of the referees towards Wyld’s work, which they said had been subject to later criticism. As is so often the case with referees in this field, they accepted the criticism as utterly damning, without apparently any critical thought, or ability to produce a nuanced reaction, on their own part.

My aim in this blog is to explain what I noticed about Wyld’s formalism all those years ago, and I shall give only as much of his method as necessary to make this a brief and understandable point. We begin with the Fourier-transformed solenoidal Navier-Stokes equation, written in an extremely compressed notation as: \begin{equation} \mathcal{L}_{0,k}u_k = \lambda M_{0,k}u_ju_{k-j},\end{equation} where the linear operator $\mathcal{L}_{0,k} = \partial /\partial t + \nu_0 k^2$, ν_0 is the kinematic viscosity of the fluid, $M_{0,k}$ is the inertial transfer operator which contains the eliminated pressure term, and λ is a book-keeping parameter which is used to keep track of terms during an iterative solution. Properly detailed versions of these equations may be found in either [3] or [4], but these will be sufficient for my present purposes.

Now let us begin with the closure problem. We multiply equation (1) through by u_{-k} and average, to obtain: \begin{equation} \mathcal{L}_{0,k}\langle u_k u_{-k}\rangle= \lambda M_{0,k}\langle u_ju_{k-j}u_{-k}\rangle,\end{equation} where the angle brackets denote an average. Then we set up a perturbation-type approach by expanding the velocity field in powers of λ as: \begin{equation} u_k = u^{(0)}_k + \lambda u^{(1)}_k + \lambda^2 u^{(2)}_k + \lambda^3 u^{(3)}_k + \dots, \end{equation} where $u^{(0)}_k$ is a velocity field with a Gaussian distribution.

The general procedure then has two steps. First, substitute the expansion (3) into the right hand side of equation (1) and calculate the coefficients iteratively in terms of the $u^{(0)}_k$. Secondly, substitute the explicit form of the expansion, now entirely expressed in terms of the $u^{(0)}$ into the right hand side of equation (2), and evaluate the averages to all orders, using the rules for a Gaussian distribution. If we denote the inverse of the linear operator by $\mathcal{L}^{-1}_{0,k} \equiv R_{0,k}$, and the Gaussian zero-order covariance by $\langle u_k u_{-k}\rangle=C_{0,k}$, then the triple moment on the right hand of equation (2) can be written to all orders in products and convolutions of $R_{0,k}$ and $C_{0,k}$.

Wyld did not follow this procedure exactly. Instead, he inverted the linear operator on the left hand side of (2), and wrote an expression for the exact covariance C_k as: \begin{equation} \langle u_k u_{-k}\rangle \equiv C_k= R_{0,k}\lambda M_{0,k}\langle u_ju_{k-j}u_{-k}\rangle.\end{equation} Of course, (4) is mathematically equivalent to (2), so does this matter? Well, when we consider renormalization, it does!

Kraichnan introduced renormalization in this problem as making the replacements: \[R_{0,k}\rightarrow R_{k} \quad \mbox{and} \quad C_{0,k} \rightarrow C_k\] to all orders in the perturbation expansion of the triple-moment in (2). When Wyld used diagram methods to show how such a renormalization could come about, by summing subsets of terms to all orders, he in effect also renormalized both the explicit operators $R_{0,k}$ and $M_{0,k}$ on the right hand side of (4). The first of these erroneous steps created the famous double-counting problem, while the second raised questions about vertex renormalization. A full account of this topic and the introduction of `improved Lee-Wyld theory' can be found in reference [5].

Lastly, for sake of completeness, I should mention that reference [2] was superseded in 2017 by reference [6], as the derivation of the two-time LET theory.

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[5] A. Berera, M. Salewski, and W. D. McComb. Eulerian Field-Theoretic Closure Formalisms for Fluid Turbulence. Phys. Rev. E, 87:013007-1-25, 2013.
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