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Vacation post No. 2: I will be out of the virtual office until Monday 19 April.

Recently, someone who posted a comment on one of my early blogs about spectral methods (see the post on 20 February 2020), commented that a certain person has said `spectral methods obscure the physics of turbulence'. They asked for my opinion on this statement and I gave a fairly robust and concise reply. However, on reflection, I thought that a more nuanced response might be helpful. As the vast majority of turbulence researchers work in real space, it seems probable that many would share that sentiment, or something very like it.

In fact, I will begin by challenging the second part of the statement. What precisely is meant by the phrase `the physics of turbulence'? In order to answer this question, let us begin by examining the concept of the turbulence problem in both real space and Fourier wavenumber space. Note that in what follows, all dependent variables are understood to be per unit mass of fluid, and we restrict our attention to incompressible fluid motion.

In real space, we have the velocity field $\mathbf{u}(\mathbf{x},t)$, which satisfies the Navier-Stokes equation (NSE). This equation expresses conservation of momentum and is local in x. It is also nonlinear and is therefore, in general, insoluble. From it we can derive the Karman-Howarth equation (KHE), which expresses conservation of energy and relates the second-order moment to the third-order moment. This is also local in x, and is also insoluble, as it embodies the statistical closure problem of turbulence. If we wish, we can change from moments to structure functions, but the KHE remains local in r, the distance between the two measuring points. This formulation gives no hint of a turbulence cascade as it is entirely local in nature.

The situation is radically different in Fourier wavenumber (k) space. Here we have a velocity field $\mathbf{u}(\mathbf{k},t)$ which now satisfies the NSE in k-space. This is still insoluble, and when we derive the Lin equation from it (or by Fourier transformation of the KHE), this again expresses conservation of energy, and is again subject to the closure problem. However, there is a major difference. As pointed out by Batchelor [1], Taylor introduced the Fourier representation in order to turn turbulence into a problem in statistical physics, with the $\mathbf{u}(\mathbf{k},t)$ playing the part of the degrees of freedom. The nonlinear term takes the form of a convolution in wavenumber space and this couples each degree of freedom to every other. In the absence of viscosity, this process leads to equipartition, rather as in an ideal gas. However, the viscous term is symmetry-breaking, with its factor of k^2 skewing its effect to high wavenumbers, so that energy must flow through the modes of the system from low wavenumbers to high. We may complete the picture by injecting energy at low wavenumbers. The result is a physical system which has been discussed in many papers and books and has been studied by theoretical physicists over the decades since the 1950s. In short, Fourier transformation reveals a physical system which is not apparent from the equations of motion in real space.

What, then, do those working in real space mean by the physics of turbulence? Presumably they rely on ideas about vortex motion, as established by flow visualisation; and here the difficulty lies. Richardson put forward the concept of a cascade in terms of `’whirls” (not, incidentally, whorls! [2]); and certainly this has gripped the imagination of generations of workers in the field. In a general, qualitative way it is easy to understand; and one can envisage the transfer of eddying motions from large scales to small scales. But when it comes to a quantitative point of view, the resulting picture is very vague and imprecise. Of course attempts have been made to make it more precise and researchers have considered assemblies of well-defined vortex motions. This is a perfectly reasonable way for fluid dynamicists to go about things, but it involves a considerable element of guess work. In contrast, Fourier wavenumber space gives a precise representation of the physical system and essentially formulates the basic problem as a statistical field theory.

So, spectral methods actually expose the underlying physics of turbulence, rather than obscuring it. It is my view that those who are not comfortable with them must necessarily have a very restricted and limited understanding of the subject. I shall illustrate that in my next post.

[1] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 2nd edition, 1971.
[2] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.