Stirring forces and the turbulence response.

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Vacation post No. I: I will be out of the office until Monday 19 April.

In my previous post, I argued that there seems to be really no justification for regarding the stirring forces that we invoke in isotropic turbulence as mysterious, at least in the context of statistical physics. However, when I was thinking about it, I remembered that Kraichnan had introduced stirring forces in quite a different way from Edwards and it occurred to me that this might be worth looking at again. Edwards had introduced them in order to study stationary turbulence, but in Kraichnan's case they were central to the basic idea for his turbulence theory. In that way, Kraichnan's formulation was more in the spirit of dynamical systems theory, rather than statistical physics.

Following Kraichnan, let us consider the case where the Navier-Stokes equation (NSE) is subject to random force $f_{\alpha}(\mathbf{k},t)$, where the Greek indices take the usual values of $1,\,2,\,3$ corresponding to Cartesian tensor notation. If the force undergoes a fluctuation \[f_{\alpha}(\mathbf{k},t) \rightarrow f_{\alpha}(\mathbf{k},t) +\delta f_{\alpha}(\mathbf{k},t),\] then we may expect the velocity field to undergo a corresponding fluctuation \[u_{\alpha}(\mathbf{k},t) \rightarrow u_{\alpha}(\mathbf{k},t) +\delta u_{\alpha}(\mathbf{k},t).\] If the increments are small enough, we may neglect the second order of small quantities, then we may introduce the infinitesimal response function $\hat{R}_{\alpha\beta}(\mathbf{k};t,t')$, such that \[\delta u_{\alpha}(\mathbf{k},t) = \int_{-

\infty}^t\,\hat{R}_{\alpha\beta}(\mathbf{k};t,t’)\delta f_{\beta}(\mathbf{k},t’)\,dt’.\]

Kraichnan linearised the NSE in order to derive a governing equation for the infinitesimal response function. Then he introduced the ensemble-averaged form \[\langle\hat{R}_{\alpha\beta}(\mathbf{k};t,t’)\rangle =R_{\alpha\beta}(\mathbf{k};t,t’),\] where \[R_{\alpha\beta}(\mathbf{k};t,t)=1,\] in order to make a statistical closure. The result was the Direct Interaction Approximation (DIA) and it is worth noting in passing that its derivation contains the step $\langle uu\hat{R} \rangle = \langle uu \rangle \langle \hat{R}\rangle$, which makes the theory a mean-field approximation.

The failure of DIA was attributed by Kraichnan to the use of an Eulerian coordinate system and he responded by generalising DIA to what he called Lagrangian-history coordinates, leading to a much more complicated formulation. This step inspired others to DIA-type methods in more conventional Lagrangian coordinates. However, the fact remains that the purely Eulerian LET (or local energy transfer) does not fail in the same way as DIA. It is worth noting that unsuccessful theories in Eulerian coordinates are invariably Markovian in wavenumber (this should be distinguished from a Markovian property in time).

An alternative explanation for the failure of Markovian theories is that the basic ansatz, in the steps outlined above, may not identify the correct response for turbulence. In dynamical systems the dissipation occurs where the force acts. In turbulence it occurs at a distance in space and time. When the force acts to stir the fluid, the energy is transferred to higher wavenumbers by a conservative process, until it comes into detailed balance with the viscous dissipation. Arguably the system response needs to include some further effect, connecting one velocity mode to another, as happens in the LET theory [1].

In all theories, the direct action of the stirring force is both to create the modes and then populate them with energy. In DIA, the way in which energy is put into the modes (i.e. the input term) can be calculated exactly by renormalized perturbation theory in terms of the ensemble-averaged response function . However, the general closure of the statistical equations for the velocity moments is equivalent to an assumption that the same procedure will work for it, which is really only an assumption. So it may be that it is the turbulence response which is mysterious, and not the stirring forces as such.

General treatments of these matters will be found in the books [2,3]. It should be noted that I've used a modern notation for the response function (e.g. see [4]).

[1] W. D. McComb and S. R. Yoffe. A formal derivation of the local energy transfer (LET) theory of homogeneous turbulence. J. Phys. A: Math. Theor., 50:375501, 2017.
[2] D. C. Leslie. Developments in the theory of turbulence. Clarendon Press, Oxford, 1973.
[3] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990
[4] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.