Analogies between critical phenomena and turbulence: 2

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In the previous post, I discussed the misapplication to turbulence of concepts like the relationship between mean-field theory and Renormalization Group in critical phenomena. This week I have the concept of '*anomalous exponents*' in my sights!

This term appears to be borrowed from the concept of anomalous dimension in the theory of critical phenomena, so we start from a consideration of dimension, bearing in mind that the dimension of the space can be anything from $d=1$ up to $d=\infty$, and is not necessarily an integer. In critical phenomena it is usual to define three different kinds of dimensionality, as follows:

[a] **Scale dimension.** This is defined as the dimension of a physical quantity as established from the effect of a scaling transformation. Confusingly, this is normally just referred to as dimension.

[b] **Normal (canonical) dimension.** This is the (scale) dimension as established by simple dimensional analysis.

[c] **Anomalous dimension.** This is the dimension as established under RG transformation.

In this context, normal dimension is regarded as the naïve dimension and anomalous dimension is regarded as the actual or correct dimension. In turbulence we don't have dimensionality as a playground, so the merry band of would-be turbulence theorists have extended the concept to the *exponents* of power-law forms of the moments of the velocity field plotted against order. The Kolmogorov forms (dimensional analysis) are seen as canonical and the actual (i.e. measured) exponents are seen as anomalous. The former are seen as wrong and the latter as correct. Naturally, the true believers in intermittency corrections have seized on this nomenclature as adding something to their case. (Also, see my post of 21 January 2021).

Let us actually apply the concept of scale dimension d_s (say) in three-dimensional turbulence (i.e. $d=3$), using the procedures from critical phenomena (see Section 9.3 of [1]) to the energy spectrum $E(k)$. That is, we express the spectrum in terms of the total energy E, thus \[\int\,d^3k\,E(k) = E \quad \mbox{hence} \quad E(k) \sim E\,k^{-3}.\] So, bearing in mind that wavenumber has dimensions of inverse length, it follows that the canonical scale dimension is $d_s = 3$ in $d=3$.

If we now consider the Kolmogorov spectrum based on scale invariance and an inertial transfer rate ε_T, dimensional analysis gives us \[E(k) \sim \varepsilon_T\,k^{-5/3} .\] As this result can also be got from RG transformation, properly formulated for macroscopic fluid turbulence, and employing rational approximations (see [2] – [5]), it follows that K41 corresponds to the anomalous dimension $d_E = 5/3$. So much for inept comparisons with critical phenomena.

[1] W. D. McComb. Renormalization Methods: A Guide for Beginners. Oxford University Press, 2004.
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[3] W. D. McComb, W. Roberts, and A. G. Watt. Conditional-averaging procedure for problems with mode-mode coupling. Phys. Rev. A, 45(6):3507-3515, 1992.
[4] W. D. McComb and A. G. Watt. Two-field theory of incompressible-fluid turbulence. Phys. Rev. A,

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[5] W. D. McComb. Asymptotic freedom, non-Gaussian perturbation theory, and the application of renormalization group theory to isotropic turbulence. Phys. Rev. E, 73:26303-26307, 2006.