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Earlier this year I received an enquiry from Alex Liberzon, who was puzzled by the fact that some people plot temporal frequency spectra with a $-5/3$ power law, but he was unable to reconcile the dimensions. This immediately took me back to the 1970s when I was doing experimental work on drag-reduction, and we used to measure frequency spectra and convert them to one-dimensional wavenumber spectra using Taylor's hypothesis of frozen convection [1]. It turned out that Alex's question was more complicated than that and I will return to it at the end. But I thought my own treatment of this topic in [1] was terse, to say the least, and that a fuller treatment of it might be of general interest. It also has the advantage of clearing the easier stuff out of the way!

Consider a turbulent velocity field $u(x,t)$ which is stationary and homogeneous with rms value U. According to Kolmogorov (1941) [2], the mean square variation in the velocity field over a distance r from a point x is given by:\begin{equation}\langle \Delta u^2_r \rangle \sim (\varepsilon r)^{2/3}.\end{equation} If we now consider the turbulence to be convected by a uniform velocity U_c in the x-direction, then the K41 result for the mean square variation in the velocity field over an interval of time τ at a point x is given by: \begin{equation}\langle \Delta u^2_\tau \rangle \sim (\varepsilon U_c\tau)^{2/3}.\end{equation}The dimensional consistency of the two forms is obvious from inspection.

Next let us examine the dimensions of the temporal and spatial spectra. We will use the angular frequency $\omega = 2\pi f$, where $f $ is the frequency in Hertz, in order to be consistent with the definition of wavenumber k_1, where k_1 is the component of the wavevector in the direction of x. Integrating both forms of the spectrum, we have the condition: \begin{equation} \int^\infty_0 E(\omega) d\omega = \int_0^\infty E_{11}(k_1) dk_1 = U^2. \end{equation} Evidently the dimensions are given by: \begin{equation}\mbox{Dimensions of}\, E(\omega)d\omega = \mbox{Dimensions of}\, E_{11}(k_1) dk_1 = L^2 T^{-2};\end{equation} or velocity squared.

Then we introduce Taylor’s hypothesis in the form: \begin{equation} \frac{\partial}{\partial t} = U_c \frac{\partial}{\partial x}, \quad \mbox{thus} \quad \omega = U_c k_1;\end{equation} and hence: \begin{equation}k_1= \frac{\omega}{U_c} \quad \mbox{and} \quad dk_1 = \frac{d\omega}{U_c}. \end{equation}
The Kolmogorov wavenumber spectrum (in the one-dimensional form that is usually measured) is given by:\begin{equation}E_{11}(k_1) = \alpha_1 \varepsilon^{2/3} k^{-5/3}_1 dk_1.\end{equation}We should note that α_1 is the constant in the one-dimensional spectrum and is related to the three-dimensional form α by $\alpha_1 = (18/55)\alpha $. Substituting for the wavenumbers from (6) into (7) we find:\begin{equation} E_{11}(k_{1})dk_{1} = \alpha_1 (\varepsilon U_c)^{2/3}\omega^{5/3} d\omega \equiv E(\omega)d\omega, \end{equation} which is easily shown to have the correct dimensions of velocity squared.

After seeing this analysis, Alex came back with: but what about when the field is homogeneous and isotropic, with $U_c=0$? That’s a very good question and takes us into a topic which originated with Kraichnan’s analysis of the failure of DIA in (1964) [1]: the importance of sweeping effects on the decay of the velocity correlation. There are now numerous papers which address this topic and they continue to appear. So it does not give the impression of being settled. From my point of view, this is important in the context of closure approximations; but I understand that the answer to the question of $f^{-5/3}$ or f^{-2} depends on the importance or otherwise of sweeping effects.

I intend to return to this, but not necessarily next week!

[1] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.
[2] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. C. R. Acad. Sci. URSS, 30:301, 1941.