Can statistical theory help with turbulence modelling?

Can statistical theory help with turbulence modelling?
When reading the book by Sagaut and Cambon some years ago, I was struck by their balance between fundamentals and applications [1]. This started me thinking, and it appeared to me that I had become ever more concentrated on fundamentals in recent years. In other words, I seemed to epitomize the old saying about scholarship consisting of `learning more and more about less and less'!

It was not always so. I began my career in research and development, which was very practical indeed. Then my employers sent me back to university where I took a degree in theoretical physics, followed by a PhD on the statistical theory of turbulence. Obviously the rot had set in; but, even so, in later years I did quite a lot of experimental work on drag reduction by additives and also turbulent diffusion. At least these topics had a practical orientation. Moreover, I have also used the $k-\varepsilon$ model to carry out calculations on the `jet in crossflow' problem. This might seem surprising, but it arose quite naturally in the following way.

Around about 1980 I had a call from a colleague in the maths department at Edinburgh. The Iran-Iraq war had recently broken out, and one of his MPhil students came from that part of the world. The student had decided that he would rather take a PhD than go home and be involved in the fighting. Very understandable, but the difficulty was that he needed a more substantial project. At present he was studying the jet in crossflow problem, using ideal flow methods. My colleague wondered if I could join in as co-supervisor and introduce some turbulence to the project in order to make it more realistic.

Lacking any experience in this field, I happily agreed to join in, and proposed that we use the $k-\varepsilon$ model, which at the time was the best known of the engineering models. We set out on a programme of studying both the model and associated numerical methods, in the process considering a hierarchy of problems of increasing difficulty, until we reached the jet in crossflow.

This was a long time ago, but two things about this PhD supervision remain in my memory. First, the student was a mathematician and had no prior knowledge of numerical computation. This leaves me with an abiding impression that he initially found it very difficult to realise that we did not need to be able to solve an equation in the mathematical sense. Because of this, we had many discussions which appeared to be going well and then ended in frustration. Secondly, once we managed to encourage him to overcome his reluctance and try to use the computer, he proved to be a natural and worked rapidly through our hierarchy of problems, ending up with useful results in a commendably short time. This happened at a time of upheaval for me, when I was moving from the School of Engineering to the School of Physics, so I have only a rather vague memory of how things turned out. I believe that he got his PhD and then went on somewhere in England as a postdoc. Whether the results were published or not, I don’t recall. But the experience left me with an appreciation of the value of a practical engineering model, where my own fundamental work would have been of little assistance. A short discussion of the $k-\varepsilon$ model can be found in Section 3.3.4 of my book, given as reference [2] below.

When considering how statistical theory might help, we should first recognize that it does give rise to a class of models, beginning with the Eddy Damped Quasi-Normal model (which is cognate to the self-consistent field theory of Edwards) and has a single adjustable constant. It is, however, restricted to homogeneous turbulence. What we could really do with is something like $k-\varepsilon$, which is a single-point theory, but which arises in a systematic way from a two-point statistical theory. The value of the latter is that it takes into account spatially (and temporally) nonlocal effects.

The details of the statistical closure theories are complicated, but the basic idea of how one might try to derive single-point engineering models is quite simple. The key quantity is the covariance of two fluctuating velocities at different points (and times) and a theory consists of a closed set of equations to determine the covariance. In general, the covariance tensor is a matrix of nine covariance functions, although symmetry will often reduce that. We will consider just one such function, which we write as $C(\mathbf{x},\mathbf{x’})$, leaving the time variables out for simplicity. We then make the change of variables to centroid and relative coordinates, thus: \[\mathbf{R} = (\mathbf{x} + \mathbf{x’})/2 \qquad \mbox{and} \qquad \mathbf{r} = (\mathbf{x} – \mathbf{x’}). \]

Now, the statistical theories are studied for the homogeneous case in order to simplify the problem. That is, we assume that there is no dependence on the centroid coordinate; and Fourier transform into wavenumber space, with respect to the relative variable. However, the basic derivation and renormalization are not restricted to this case, and we can write down equations for the general case. Then, recognizing that most turbulent shear flows have a smooth dependence on the centroid coordinate, we can envisage expanding in the centroid coordinate, with coefficients obtained as integrals over wavenumber. Then, setting $x=x’$, we could end up with single-point equations, where coefficients are determined by integrals that arise in the fundamental theory.

This would not be a trivial process but, given the huge importance of turbulence calculations in a variety of applications, it is perhaps surprising that it has been so comprehensively neglected. A recent discussion of statistical two-point closures can be found in reference [3]. For completeness, I should mention that a second edition of [1] has appeared and I understand that a third edition is in the pipeline.

[1] P. Sagaut and C. Cambon. Homogeneous Turbulence Dynamics. Cambridge University Press, Cambridge, 2008.
[2] W. D. McComb. The Physics of Fluid Turbulence. Oxford University Press, 1990.
[3] W. D. McComb and S. R. Yoffe. A formal derivation of the local energy transfer (LET) theory of homogeneous turbulence. J. Phys. A: Math. Theor., 50:375501, 2017.