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In the preceding two posts, we have pointed out that the final statement by Onsager in his 1949 paper [1] is, in the absence of a proper limiting procedure, only a conjecture; and that the infinite Reynolds number limit, as introduced by Batchelor [2] and extended by Edwards [3], shows that it is incorrect. We have also shown that it is not in accord with the way in which turbulence is nowadays known to behave. In this post, we consider the question of how well the Batchelor-Edwards picture of dissipation agrees with the experimental picture and consider the nature of the equations of fluid motion. Additionally, we consider the physical nature of the process that we term 'dissipation'.

A particular problem with Onsager's paper is that it conflates two quite distinct situations. These are: the infinite Reynolds number limit, on the one hand; and the breakdown of the continuum limit, on the other. In order to distinguish between these two, we have to distinguish between the two kinds of Navier-Stokes equation (NSE). If we wish to take a true (in the mathematical sense) infinite Reynold number limit, then we must work with the equations of continuum mechanics. If we want to consider the breakdown of the continuum limit, then we must consider a fluid made of molecules, in which the equations of motion were derived by a macroscopic averaging process. I have touched on this distinction in my post of 14 May 2020 and will develop it in rather more detail here.

The equations of fluid motion, as they are normally encountered by engineers and applied mathematicians, are derived macroscopically; and rely on the concept of a fluid continuum which is without structure. They express Newton's second law of motion as applied to the continuum and are based on a linear approximation to the relation between the shear stresses and corresponding rates of strain. Fortuitously, this approximation applies to a wide class of fluids. They are also based on the assumption of incompressibility which means that the macroscopic fluid motions do not produce density changes. Of course, sound waves will travel through any fluid, so strictly the incompressibility is only an approximation.

The NSE is expected to describe the macroscopic motion of any Newtonian fluid. If we set the viscosity equal to zero, then we have the Euler equation, which is taken to apply to an *ideal fluid*. It then provides a relationship between velocity and pressure for fluid motions which are remote from boundaries. Combined with the concept of streamline flow, it leads to the *Bernoulli equation*. This can be solved for practical problems by the use of *ad hoc* coefficients which take effects such as viscosity into account. Also, the Euler equation can be combined with boundary layer theory to describe real fluid motions.

If we consider the Batchelor-Edwards infinite Reynolds number limit [2,3], which locates the dissipation at $k=\infty$, then this can only apply in the continuum mechanics picture just outlined. What, then, is the use of such a limit? The answer is that it is useful in any context where one's theory is based on the continuum model. In the case of Edwards, he applied it to his self-consistent field theory of turbulence. Of course, as we pointed out in the preceding post, this is mathematically equivalent to Kraichnan's use of scale-invariance in testing his direct-interaction approximation.

Now let us turn to the microscopic derivation of the NSE. This begins at the molecular level and one ends up by averaging over volumes which are small compared to the flow volume but large enough to contain very large numbers of molecules. Evaluating such averages is seen as a limiting process and is often referred to as the *continuum limi*t.

It is worth quoting what Batchelor said (ibid page 5) after discussing the possibility that small scale motions might not satisfy the continuum limit, he went on: ‘However, the action of viscosity is to suppress strongly the small-scale components of turbulence and we shall see that for all practical conditions the spectral distribution of energy dies away effectively to zero long before length scales comparable with the mean free path are reached. As a consequence, we can ignore the molecular structure of the medium and regard it as a continuous fluid.’

In my post on 14 May 2020, I quoted a calculation by Leslie [4], making exactly the same point, but in a more quantitative way. As an aside, I note that over the years I have heard many speculations about singularities and near-singularities (*sic*), but I have never heard of anyone making such speculations actually doing a calculation to establish under just what circumstances this pathological behaviour might be expected to occur. As we have seen, and will discuss further in our forthcoming paper [5], the practical onset of scale-invariance is at quite a moderate Reynolds number.

We will conclude by considering what we mean by ‘viscous dissipation’. This is the rate at which the kinetic energy of fluid motion is randomised at the molecular level, with the result that the fluid heats up. Turbulent dissipation is of course known to be very much larger, but the turbulent motions are themselves dissipated by molecular motion and again the fluid heats up. This is a two-stage process, with energy being transferred through wavenumber until it is finally dissipated by viscosity. As the Reynolds number increases, the volume of wavenumber space also increases, such that a greater amount of energy can be accommodated, and this leads to scale-invariance, and to apparent independence of the coefficient of viscosity. This absorption of energy may be seen as a *quasi-dissipation* but the real dissipation still happens at the end of the cascade! It would be really quite strange if this limiting process led to a situation where there was only quasi-dissipation and the fluid no longer heated up. In other words, if the Onsager view were to prevail over the Batchelor-Edwards view.

[1] L. Onsager. Statistical Hydrodynamics. Nuovo Cim. Suppl., 6:279, 1949.

[2] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 2nd edition, 1971.

[3] S. F. Edwards. Turbulence in hydrodynamics and plasma physics. In Proc. Int. Conf. on Plasma Physics, Trieste, page 595. IAEA, 1965.

[4] D. C. Leslie. Developments in the theory of turbulence. Clarendon Press, Oxford, 1973.

[5] W. D. McComb and S. R. Yoffe. The inifinite Reynolds number limit and the quasi-dissipative anomaly. (In preparation: 2020)