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A pioneering paper on turbulence by Onsager, which was published in 1949 [1], seems to have had a profound influence on some aspects of the subject in later years. In particular, he put forward the idea that as the turbulence was still dissipative in the limit of infinite Reynolds numbers (or zero viscosity) it implied that the Euler equation must be dissipative despite its lack of viscosity. This supposed behaviour has come to be referred to as the *dissipation anomaly*. This view of matters is at odds with that of Batchelor [2] and of Edwards [3]: for a discussion see my post on 23 April 2020; but for the moment I will focus on the last paragraph in [1].

The key point involved is that the inertial-transfer term $T(k)$ of the Lin equation conserves energy, thus: $$\int_0^\infty \,dk T(k) = \int_0^\infty \, dk \int_0^\infty \, dj S(k,j) =0,$$ because of the anti-symmetry of $S(k,j)$ under interchange of k and j. Onsager uses the symbol $Q(k,k')$ for this quantity, and states the antisymmetric property as his equation (17). Once he has set the viscosity equal to zero, he concludes that the anti-symmetry of S (or his Q) no longer implies overall energy conservation. The final sentence of his paper reads: ‘The detailed conservation of equation (17) does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected (*i.e. for zero viscosity*), and the double sum of $Q(k,k')$ converges only conditionally.’ Note that the parenthesis in italics has been added by me.

Now this is open to two immediate criticisms. First, setting the viscosity equal to zero and replacing the NSE by the Euler equation, is *not* the same thing as taking the limit of zero viscosity, as done by Batchelor [2] and Edwards [3]. Secondly, the idea of ‘steps in the cascade’, although intuitively very attractive, is not sufficiently well-defined to be suitable for quantitative purposes. In contrast, the limiting process followed by Edwards is mathematically well defined and shows that in the limit of zero viscosity, the NSE possesses dissipation in the form of a delta function at $k=\infty$. Accordingly Onsager’s final statement is without justification and, on the Batchelor-Edwards picture, is incorrect.

These arguments deal with extreme situations, but a more moderate approach is to follow the second method of defining the infinite-Reynolds number limit, which also arises out of Batchelor’s work and which leads to the concept of *scale-invariance* of the inertial flux. This approach was followed by Kraichnan and many others; and, although differing in detail, is mathematically equivalent to the Edwards formulation. We will discuss this in the next post.

[1] L. Onsager. Statistical Hydrodynamics. Nuovo Cim. Suppl., 6:279, 1949.
[2] G. K. Batchelor. The theory of homogeneous turbulence. Cambridge University Press, Cambridge, 2nd edition, 1971.
[3] S. F. Edwards. Turbulence in hydrodynamics and plasma physics. In Proc. Int. Conf. on Plasma Physics, Trieste, page 595. IAEA, 1965.