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The Gaussian, or normal, distribution plays a key part in statistical field theory. This is partly because it is the only functional which can be integrated and partly because Gaussian distributions are frequently encountered in microscopic physics at, or near, thermal equilibrium. The latter is not the case in turbulence. Indeed the non-Gaussian nature of the turbulence probability functional (pdf) is inescapable. In the absence of a mean flow, the statistical closure problem amounts to how one expresses the third-order moment $\langle uuu \rangle$ in terms of the second-order moment $\langle uu \rangle$. It is a matter of symmetry (so that it can be determined by inspection) that the third-order moment vanishes when evaluated against a Gaussian pdf. Of course various turbulence pdfs are seen to be quite close to Gaussian in form. This is particularly so for the distribution of the velocity at a single point. But some deviation from normality for a turbulence pdf is of the essence.

We will not discuss the properties of Gaussian forms here: a pedagogic treatment can be found in Appendix B of my recent book, which is cited below as reference [1]. Our aim is to give a brief discussion of three ways in which Gaussians are used in turbulence, one in Direct Numerical Simulation (DNS) and two in statistical theory. From these considerations we should be able to make a number of general points without going through a lot of complicated theory. The one theoretical aspect we should keep in mind, is the form of the solenoidal Navier-Stokes equation in wavenumber space, which we can write in a very symbolic form as: $$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_k = M_k u_ju_j + f_k .$$ Here k and j are combined wavenumbers and tensor indices, ν is the kinematic viscosity, M_k is the inertial transfer operator, u_k is the Fourier transform of the velocity field, and f_k is a stirring force, if required. A full discussion of this equation can be found in reference [1]. As ever, repeated indices are summed.

The two standard problems in DNS are (a) free decay; and (b) forced, stationary turbulence. In both cases, we start with an arbitrary (non-turbulent) velocity field, which is random and has a multivariate normal (i.e. Gaussian) distribution. The arbitrary initial energy spectrum $E(k,0)$ is chosen to be confined to very low wavenumbers. As time goes on, the nonlinear coupling in the NSE generates a velocity field at ever higher wavenumbers. In spectral terms, this is seen as $E(k,t)$ spreads out to higher wavenumbers and the skewness $-S$ rises from $S=0$ (corresponding to a Gaussian pdf) to $-S\sim 0.5$, corresponding to developed turbulence. A brief introduction to DNS may be found in Section 3.2 of reference [2].

The theoretical approach began with *quasi-normality* in the 1950s, in which one assumes that the fourth-order moment can be factorised as if Gaussian, in order to solve the second equation of the statistical hierarchy for the third-order moment. This, as is well known, led to a catastrophe. The first real advance was due to Kraichnan [3] and followed by Wyld [4], in what is now known as *renormalized perturbation theory*. In some ways, this is rather like the DNS, in that we start with a random Gaussian velocity field with a prescribed spectrum which is confined to low wavenumbers. Then, instead of stepping this forward in time on the computer, we substitute it into the non-linear term of the NSE. Assigning a book-keeping parameter λ (where $\lambda = 1$) to the nonlinear term, we expand out in powers of λ, with coefficients in the series being calculated iteratively. This is not strictly speaking perturbation theory, as λ is not small, but it resembles it, hence the name. Of course we cannot truncate at low order in λ, so we must sum infinite series, or rearrange into sub-series which can be summed. This approach leads to remarkably successful results, although there are still some questions to be answered.

The last approach was due to Edwards [5] and is the method of the self-consistent field. In this theory, Edwards used the NSE to derive a Liouville equation for the turbulence pdf. *The Gaussian pdf in this work is quite different from the other two. It is chosen to give the correct value of the two-velocity moment*. Its role then is as a *basis function* for an iterative solution of the Liouville equation as an operator-product expansion about the Gaussian zero-order distribution. Symmetry arguments play an important part in this work and if you wish to pursue this point, you will find a discussion (and an extension to two-time forms) in reference [6]. It is worth noting two points. First, this is an expansion for the exact pdf about a Gaussian and, as I remarked earlier, turbulence pdfs can be quite close to Gaussian in form. Hence there is a possibility of establishing a second-order truncation as a rational approximation. Secondly, the statistical closures derived this way are cognate to Kraichnan’s closures which are derived by very different methods. These points should encourage you to take a `glass full’ rather than a `glass empty’ view of statistical turbulence theory!

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