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When I was preparing last week’s post, I consulted the Saffman lectures in order to find an example of the culture clash between theoretical physics and applied maths. In the process I noticed quite a few points that I felt tempted to write about and in particular that old perennial question: is turbulence a single universal phenomenon? Or, does it depend on the physical situation under consideration and its conditions of formation? Over the years this question has been put numerous times by various people, both in discussions and in writing, but never seems to lead anywhere. Saffman pointed out that the opposite extreme would be to consider each situation of practical importance and describe it to the required degree of detail. At the same time he conceded that there was evidence for universality, but suggested that there might be merit in some form of cataloguing and classifying of flows.

Of course there has always been some degree of classification, even just for pedagogic purposes. For instance, free shear flows *versus* wall-bounded flows; but presumably Saffman was thinking in terms of something more profound. So far as I know, no such scheme exists; but, if it did, it might be analogous to the idea of universality classes in the theory of critical phenomena. Such phenomenon are characterised by the way macroscopic observables, such as specific heat, magnetic susceptibility or the correlation length, behave as a system tends to the critical point. They either diverge (become infinite) or go to zero. This behaviour is represented by a power-law dependence on the reduced temperature, with the introduction of critical exponents which are either positive or negative, according to the observed behaviour at the critical point. If two different physical phenomena are found to have the same values of their critical exponents, then they are said to be in the same universality class. This is of course, a purely phenomenological approach, but it corresponds to an underlying symmetry in the Hamiltonian of the system, along with the dimensionality of the space. An introductory account of this topic can be found in the book cited as reference [1] below.

There is no doubt that many of the pioneers of turbulence theory, viz. Taylor, Kolmogorov, Batchelor and Townsend, thought in terms of a correspondence between turbulence and statistical mechanics. As we have pointed out elsewhere (see Section 12.5 of reference [2]), Batchelor wrote about 'an ultimate statistical state of the turbulence' that would follow from a 'whole class of different initial conditions'. As we have also pointed out (*ibid*), one problem with this is the very great difference in the number of degrees of freedom N between the two. In effect, for canonical statistical mechanics, N is so large that fluctuations can effectively be neglected and the average and instantaneous probability distribution functions are virtually identical. This is certainly not the case for turbulence. So perhaps one can only expect a somewhat limited correspondence between the two. This is not an argument for giving up the analogy. Merely a plea for realism in employing it.

The basic idea underpinning the statistical picture of turbulence is that, as energy transfer proceeds from large eddies to small, information is lost about the conditions of their formation. Although many people prefer to think in terms of real space and 'eddies', the idea of an energy cascade is not well defined unless one works in wavenumber space, where the Fourier modes are the degrees of freedom. So, strictly speaking, one should express this in terms of transfer from small wavenumber modes to those at large wavenumber, where turbulent kinetic energy is converted into heat. This process is in accordance with the Lin equation, whereas the Karman-Howarth equation is entirely local and can tell us nothing about it.

The scaling of spectra from a variety of flows on Kolmogorov variables supports this picture and, even if there are results that do not, this does not invalidate the correctness of Kolmogorov scaling for certain flows. The valid (and interesting) question then is: how do such flows differ from those that do? A consideration of spatial symmetry may shed some light on this.

Suppose, for a simple example, we consider turbulent shear flow in the x-direction, between infinite parallel plates situated at $y=\pm a$. The flows are homogeneous in the z-direction, while the mean velocity U depends on y. If the plates are at rest, then $U(y)$ is symmetric under the interchange of y and $-y$. However, if the plates are in relative motion (Couette flow) then $U(y)$ is antisymmetric under this interchange. The first case is an approximation to flow in a plane duct (or even, with some adjustment, to pipe flow) and it is well known that Kolmogorov scaling is observed. What happens in the Couette case, I don’t know. But it would be interesting to find out. The appropriate tool for cataloguing flows in this way is to transform to centroid and difference coordinates, and make an expansion in the centroid coordinate in Taylor series. Well, it’s an idea!

Lastly, for the theoretical physicist the problem posed by the Navier-Stokes equation in wavenumber space, and driven by random noise, is a well-posed problem. It should be noted that the pioneers in this area were careful to set it up such that it could satisfy the Kolmogorov conditions for an inertial range, and in doing this they were guided by the statistical treatment of other dynamical problems, such as Brownian motion. Nowadays it is seen as belonging to a wide class of driven diffusion equations with particular relevance to soft condensed matter. Recently we have even found the surprising result that it can undergo a phase transition at low Reynolds numbers [3], so there is much still to understand about this stochastic dynamical system.

[1] W. D. McComb. Renormalization Methods. Oxford University Press, 2004.
[2] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.
[3] W. D. McComb, M. F. Linkmann, A. Berera, S. R. Yoffe, and B. Jankauskas. Self-organization and transition to turbulence in isotropic fluid motion driven by negative damping at low wavenumbers. J. Phys. A Math. Theor., 48:25FT01, 2015.