

The modified Lin equation.

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In my post of 27 February I discussed the importance of being aware of the full form of the Lin equation as this reveals the existence of a cascade in wavenumber space. In this post I want to take this a bit further, using my resolution of the *scale-invariance paradox* [1].

For me this topic first arose during a meeting in 1991 at MSRI, Berkeley. When I had finished my talk, Bob Kraichnan came up to me with a copy of my recently published book and pointed out Figure 2.5, which was a plot of the terms in the Lin equation for freely decaying turbulence. He commented on the fact that the transfer spectrum $T(k)$ was shown as zero for an extended range of values of k . He commented that people used to think that was the case, because it would be expected from the scale-invariance of the flux, but that in practice it was never observed. There was always a single zero-crossing. I was able to reassure him that figure was based on a computation of the LET theory; that there had been an error which had now been rectified; and that the revised figure would show a single zero-crossing and would appear in the paperback edition of the book to be published later that year.

However, I was left with a nagging feeling that there was an unresolved problem with this result. The first measurements of $T(k)$ had been published by Uberoi [2] in 1963, and this author had said that the single zero-crossing was probably due to the low Reynolds number and indicated that he would expect $T(k)=0$ over an extended range of k to develop with increasing Reynolds number. Although this does not seem to have been a matter of widespread concern, over the 1970s/80s/90s various ad hoc methods were used to cope with this behaviour in numerical calculations: for some references

to this work, see [1]. As a matter of interest, I include both versions of Figure 2.5 below.

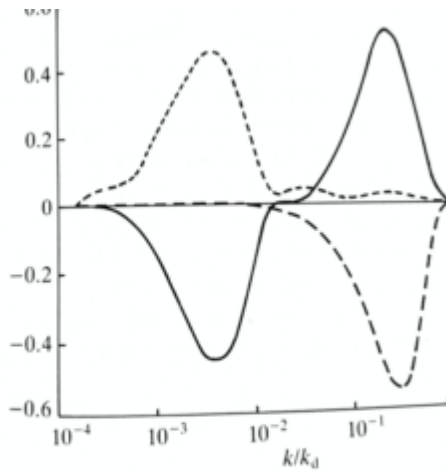


Fig. 2.5. Sketch of the three-dimensional energy, dissipative isotropic turbulence: (— $T(k, t)$; - - - $-\partial E(k, t)/\partial t$; multiplied by a factor $(k/k_d v^3)$, where k_d and v are the wavenumber and velocity scale, respectively).

Figure 2.5 from Physics of Fluid Turbulence 1990

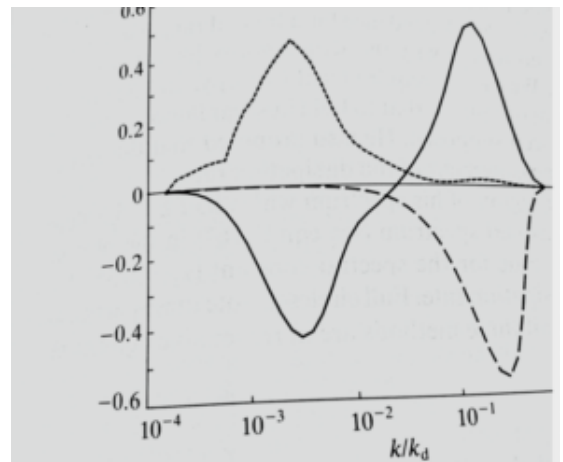


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The Lin equation (see reference [3]) takes the form:

$$\left(\frac{d}{dt} + 2 \nu k^2 \right) E(k,t) = T(k,t) \quad \text{(\ref{enbalt})}$$

where $E(k,t)$ is the energy spectrum, $T(k,t)$ is the energy transfer spectrum and ν is the kinematic viscosity. Now let us integrate each term of (\ref{enbalt}) with respect to wavenumber, from zero up to some arbitrarily chosen wavenumber κ :

$$\frac{d}{dt} \int_0^\kappa dk, E(k,t) = \int_0^\kappa dk, T(k,t) - 2 \nu \int_0^\kappa dk, k^2 E(k,t). \quad \text{(\ref{fluxbalt1})}$$

The energy transfer spectrum may be written as

$$T(k,t) = \int_0^\infty dj, S(k,j;t), \quad \text{(\ref{ts})}$$

where, as is well known, $S(k,j;t)$ can be expressed in terms of the triple moment. Its antisymmetry under interchange of k and j guarantees energy conservation in the form:

$$\int_0^\infty dk, T(k,t) = 0. \quad \text{(\ref{encon})}$$

With some use of the antisymmetry of S , along with equation (\ref{encon}), equation (\ref{fluxbalt1}) may be written as

$$\frac{d}{dt} \int_0^\kappa dk, E(k,t) = - \int_0^\infty dk, \int_0^\kappa dj, S(k,j;t) - 2 \nu \int_0^\kappa dk, k^2 E(k,t). \quad \text{(\ref{fluxbalt2})}$$

the integral of the transfer term is readily interpreted as the net flux of energy

from wavenumbers less than κ to those greater than κ , at any time t .

It is convenient to introduce a specific symbol Π for this energy flux, thus:
$$\Pi(\kappa, t) = \int_{-\infty}^{\infty} dk \, T(k, t) = - \int_{\kappa_0}^{\kappa} dk \, T(k, t),$$
 where the second equality follows from (encon).

The key to resolving the paradox is to introduce transfer spectra which have been filtered with respect to k and which have had their integration over j partitioned at the filter cut-off, i.e. $j = k_c$ [1],[4]. Beginning with the Heaviside unit step function, defined by:

$$H(x) = \begin{cases} 1 & \text{for } x > 0; \\ 0 & \text{for } x < 0. \end{cases}$$
 we may define low-pass and high-pass filter functions, thus:

$$\theta^-(x) = 1 - H(x)$$
 and
$$\theta^+(x) = H(x).$$
 We may then decompose the transfer spectrum, as given by (ts),

into four constituent parts,
$$T^-(k|k_c) = \theta^-(k - k_c) \int_{\kappa_0}^{\infty} dj \, S(k, j);$$

$$T^{+-}(k|k_c) = \theta^-(k - k_c) \int_{-\infty}^{\kappa_0} dj \, S(k, j);$$

$$T^{+-(k|k_c) = \theta^+(k - k_c) \int_{\kappa_0}^{\infty} dj \, S(k, j);$$

$$T^{++}(k|k_c) = \theta^+(k - k_c) \int_{-\infty}^{\kappa_0} dj \, S(k, j),$$

such that the overall requirement of energy conservation is satisfied:

$$\int_{\kappa_0}^{\infty} dk \left[T^-(k|k_c) + T^{+-}(k|k_c) + T^{+-(k|k_c) + T^{++}(k|k_c) \right] = 0.$$
 It is readily verified that the individual filtered/partitioned transfer spectra have the following properties:

$$\int_{\kappa_0}^{\infty} dk \, T^-(k|k_c) = 0;$$

$$\int_{\kappa_0}^{\infty} dk \, T^{+-}(k|k_c) = -\Pi(k_c);$$

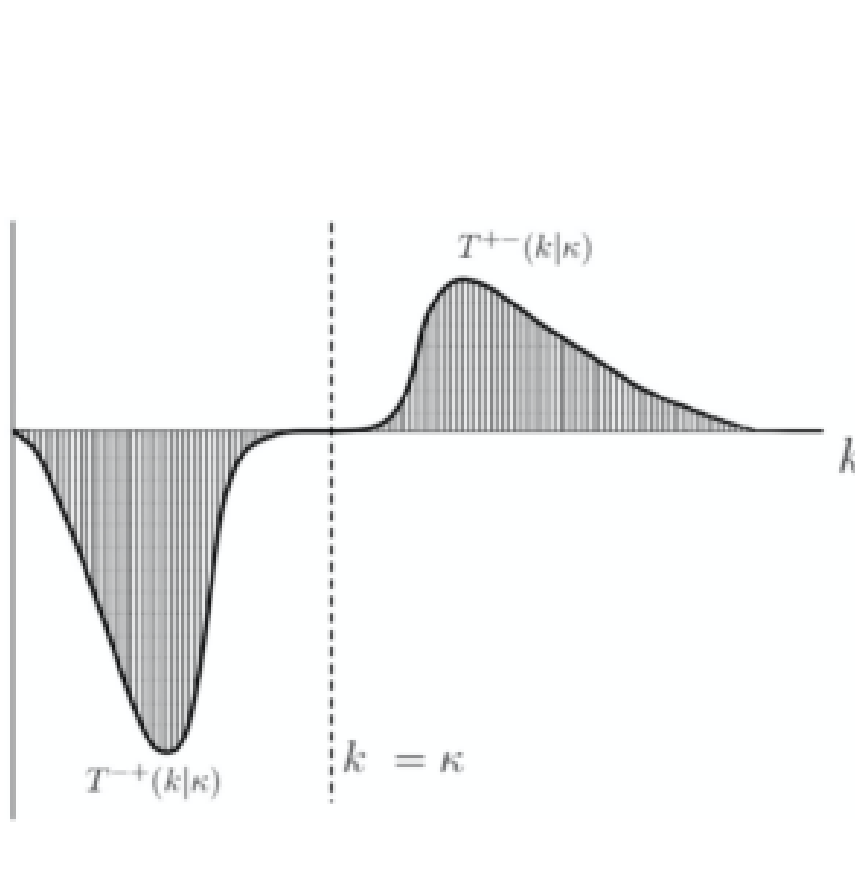
$$\int_{k_c}^{\infty} dk, T^{+-}(k|k_c) = \Pi(k_c); \quad \text{\label{pm}}$$
 and

$$\int_{k_c}^{\infty} dk, T^{++}(k|k_c) = 0. \quad \text{\label{pp}}$$
 Equation (\ref{fluxbalt1}) may be rewritten in terms of the filtered/partitioned transfer spectrum as:

$$E(k,t) = -\int_{k_c}^{\infty} dk, T^{+-}(k|k_c) - 2\nu_0 \int_{k_c}^{\infty} dk, k^2 E(k,t). \quad \text{\label{fluxbaltmod}}$$
 We note from equation (\ref{mm}) that $T^{-}(k|k_c)$ is conservative on the interval $[0, k_c]$, and hence does not appear in (\ref{fluxbaltmod}), while $T^{-+}(k|k_c)$ has been replaced by $-T^{+-}(k|k_c)$, using (\ref{mp}) and (\ref{pm}). Those working with DNS or analytical theory, can avoid the paradox by changing their definition of energy fluxes, from those given by (\ref{tp}), to the forms:

$$\Pi(\kappa, t) = \int_{\kappa}^{\infty} dk, T^{+-}(k|\kappa, t) = -\int_{\kappa}^{\infty} dk, T^{-+}(k|\kappa, t), \quad \text{\label{tpmod}}$$
 where $T^{+-}(k|\kappa, t)$ is defined by (\ref{tpm}) and $T^{-+}(k|\kappa, t)$ by (\ref{tmp}). This is equivalent to (\ref{tp}); but, unlike it, avoids the paradox.

This behaviour is illustrated in the figure below, where we should note that $T^{-+}(k|\kappa)$ is defined below the cut-off wavenumber $\kappa = k_c$, and $-T^{+-}(k|\kappa)$ is defined above it.



Modified form of transfer spectrum to avoid the scale-invariance paradox.

This raises the question of how exactly the Lin equation should be written, in order to emphasise these properties. That will be the subject of a paper which is now in preparation [5]. It is worth making the point that the filtered-partitioned forms of the transfer spectrum have only been studied in the context of the subgrid modelling problem [4]. Given the much more powerful computers now available, it would undoubtedly be rewarding to study the role of these terms in the energy balance for a range of Reynolds numbers. I very much hope that someone will do this.

Acknowledgement: the above figure was suggested by John Morgan, who also prepared it.

[1] David McComb. Scale-invariance in three-dimensional turbulence: a paradox and its resolution. J. Phys. A: Math.

Theor., 41:75501, 2008.

[2] M. S. Uberoi. Energy transfer in isotropic turbulence. *Phys. Fluids*, 6:1048, 1963.

[3] W. David McComb. *Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures*. Oxford University Press, 2014.

[4] W. D. McComb and A. J. Young. Explicit-Scales Projections of the Partitioned Nonlinear Term in Direct Numerical Simulation of the Navier-Stokes Equation. In *Proc. 2nd Monte Verita Colloquium on Fundamental Problematic Issues in Turbulence*: available at [arXiv:physics/9806029](https://arxiv.org/abs/physics/9806029) v1, 1998.

[5] W. D. McComb. A modified Lin equation for the energy balance in isotropic turbulence. [arXiv:2007.13622v1](https://arxiv.org/abs/2007.13622) [physics.flu-dyn] 27 Jul 2020