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When I was first publishing, in the early 1970s, referees would often say something like `the author uses the *turbulence in a box concept*' before going on to reveal a degree of incomprehension about what I might be doing, let alone what I actually was doing. A few years later, when direct numerical simulation (DNS) had got under way, that phrase might have had some significance; and indeed its use is now common, albeit qualified by the word `periodic'. Of course, when Fourier methods were introduced by Taylor in the 1930s, it was in the form of Fourier series. But by the 1960s it was becoming usual among theorists to briefly introduce Fourier series and then take the infinite system limit and turn them into Fourier transforms: or, increasingly, just to formulate the problem straightaway in the infinite system. However, it can be worth one's while starting with the finite cubic box of side L, and thinking in terms of the basic physics, as well as the Fourier methods.

In order to represent the velocity field in terms of Fourier series, we introduce the wavevector $$\mathbf{k}=(2\pi/L)\{n_1,n_2,n_3\},$$ where the integers n_1,n_2,n_3 all lie in the range from $-\infty$ to ∞. Fourier sums are taken over the discrete values of $\mathbf{k}$. Then the transition to the continuous, infinite system is made by taking the limit of infinite system size, such that $$\lim_{L\rightarrow \infty}\left(\frac{2\pi}{L}\right)^3\sum_{\mathbf{k}} = \int d^3\,k.$$ As ever in physics, we assume that everything is well-behaved; and that both the field variables and their transforms exist, being independent of system size as we go to this limit.

We do not have to restrict these ideas to the Fourier representation. They are generally true when we make the transition from classical mechanics to continuum mechanics. To do this, we begin with a finite system and replace discrete objects by densities. A continuous (or field) representation is introduced by defining continuous densities in the limit of infinite system size. All physical observables must be expressed in terms of densities or rates. They cannot depend on the size of the system, otherwise we would be unable to take the continuum limit. So, if we formulate turbulence in real space in terms of structure functions in a box, then theoretical expressions for the structure functions (or equivalently, the moments) must not depend on the size of the box. This provides us with a basic first test for any theory; and to our knowledge there have been some surprising failures to recognise this. We will come back to two specific examples presently. First we will look at the general question of how to test theories.

Now, stationary isotropic turbulence can be rigorously formulated as a mathematical problem, where `rigour' is taken to be in the sense of theoretical physics, but it does not occur in nature or indeed in the laboratory. It is true that it may occur to a reasonable approximation in geophysical and astronomical flows, but at the moment it seems that DNS might be our best bet for testing mathematical theories of isotropic turbulence. So it behoves us to examine the question: how representative is DNS of the mathematical problem that we are studying?

Well, of course DNS has been an active field of research for several decades now and this aspect has not been neglected. Nevertheless, one is left with the impression that it is very much a pragmatic activity, governed by `rule of thumb' methods. For instance, when we began DNS at Edinburgh in the 1990s, I asked around for advice on the maximum value of the wavenumber that we should use, as this seemed to vary from less than the Kolmogorov dissipation wavenumber to very much greater. The consensus of advice that I received was to choose $k_{max} = 1.5 k_d$, and this is what we did. Later on, in 2001, we demonstrated a rational procedure for choosing k_{max}: see Figure 2 of reference [1] or Figure 1.6 of reference [2]. One conclusion that emerges from this, is that to resolve the dissipation rate might mean devoting one's entire simulation to the dissipation range of wavenumbers!

In recent years there seems to have been more emphasis on resolving the largest scale of the turbulence, although much of this work has been for the case of free decay. But concerns remain, particularly in the terms of experimental error. It is also necessary to note a fundamental problem. The mere fact of representing the continuum NSE on a discrete lattice is symmetry breaking for Galilean invariance and isotropy, to name but two. I'm not sure how one can take this into account, except by considering a transition towards the continuum limit and looking for asymptotic behaviour. This could involve starting with a `fully resolved' simulation and looking at increasingly finer mesh sizes. To say the least this would be very expensive in terms of computer storage and run time. Naturally, workers in the field always want the highest possible Reynolds number. But, if you begin with low Reynolds numbers, it is cheap and easy to do, and you can learn something from the variation of observables with Reynolds number. There exist some well-known simulations that have employed vast resources to achieve enormous Reynolds numbers and yet provide only a few spot values without any error bars, with no indication of asymptotic behaviour, and I understand suspicions about how well-resolved they are. An awful warning to us all!

Lastly, two more awful warnings. First, as we discussed in the previous post, Kraichnan's asymptotic solution of DIA depends on the largest scale of the system. That in itself is enough to rule it out as unphysical, whether one accepts Kolmogorov

(1941) or not. However, as I pointed out, our computations at Edinburgh do not support this asymptotic form, which was obtained analytically using approximations that Kraichnan found plausible. A critical examination of that analysis is in my opinion long overdue.

Secondly, we have the Kolmogorov (1962) form of the energy spectrum, which also depends on the largest scale of the system. Probably few people now take this work seriously, but its baleful presence influences the turbulence community and lends credence to the increasingly unrealistic idea of intermittency corrections. In fact it has recently been shown that the inclusion of the largest scale destroys the widely observed scaling on Kolmogorov variables [3]. This should have been obvious, without any need to plot the graphs!

[1] W. D. McComb, A. Hunter, and C. Johnston. Conditional mode-elimination and the subgrid-modelling problem for isotropic turbulence. Phys. Fluids, 13:2030, 2001.

[2] W. David McComb. Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures. Oxford University Press, 2014.

[3] W. D. McComb and M. Q. May. The effect of Kolmogorov (1962) scaling on the universality of turbulence energy spectra. arXiv:1812.09174[physics.fluid-dyn], 2018.