The infinite-Reynolds number limit: a first look

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I notice that MSRI at Berkeley have a programme next year on math problems in fluid dynamics. The primary component seems to be an examination of the relationship between the Euler and Navier-Stokes equations, `in the zero-viscosity limit'. The latter is, of course, the same as the limit of infinite Reynolds numbers, *providing that the limit is taken in the same way with the same constraints.* I think that it is a failure to appreciate this proviso that has resulted in the concept becoming something of a vexed question over the years. Yet it was clearly explained by Batchelor in 1953 and elegantly re-formulated by Edwards in 1965. As a result, a group of theorists has been quite happy about the concept, but many other workers in the field seem to be uneasy.

I first became aware of this when talking to Bob Kraichnan at a meeting in 1984. When I used the term, his reaction surprised me. He began to hold forth on the subject. He said that people were `frightened' of the idea of the infinite-Reynolds number limit. Rather defensively I said that I wasn't frightened by it. His reply was. `Oh, I know that you aren't but you would be surprised at the number of people who are!' Since then I have indeed been surprised by how often you get a comment from a referee which goes something like: `The authors take the infinite-Re limit … but of course you cannot really have zero viscosity, can you.' This rather nervous addendum suggests strongly that the referee does not understand the concept of a limit.

Well, one thing I would claim to understand is the idea of a limit in mathematical analysis. This is because the first class of my school course on calculus dealt with nothing else. I can remember that class period clearly, even although it was about sixty five years ago. One example that our maths master gave, was to imagine that you were cutting up your twelve-inch ruler, which was standard in those days. You cut it into two identical pieces in a perfect cutting process, with no waste. Then you put one piece over to your right hand side, and now cut the left hand piece into two identical pieces. One of these you put over to the right hand side, and add it on to the six-inch piece already there, to make a nine-inch ruler. The remaining piece you again cut into two, and move half over to make a ten and a half inch ruler. However much you repeat this process, the ruler will approach but never reach twelve inches again. In other words, twelve inches is the limit and you can only approach it asymptotically.

Suppose we carry out a similar thought experiment on turbulence; although you could actually do this, most readily by DNS. What we are going to do is to stir a fluid in order to produce stationary, isotropic turbulence. Now at this stage, we don’t even think about dissipation. We are trying to drive a dynamical system and we start by specifying the forcing in terms of the rate of doing work on the fluid. We call this quantity ε_W and it is fixed. Next our dynamical system is fully specified once we choose the boundary conditions and the kinematic viscosity ν. Accordingly, providing the forcing spectrum is peaked near the origin in wavenumber space, and there has been an appropriate choice of value of the initial kinematic viscosity, energy will enter the system at low wavenumbers, be transferred by conservative inertial processes to higher wavenumbers, and ultimately dissipated at the highest excited wavenumbers. Once the system becomes stationary, the dissipation rate must be equal to the rate of doing work, and so the Kolmogorov dissipation wavenumber is given by $k_d = (\varepsilon_W /\nu^3)^{1/4}$.

Now let us carry out a sequence of experiments in which ε_W remains fixed, but we progressively reduce the value of the kinematic viscosity. In each experiment, the viscosity is smaller and the dissipation wavenumber is larger. Therefore there is a greater volume of wavenumber space and it will take longer to fill with energy. Ultimately, corresponding to the limiting case, we have an infinite volume of wavenumber space and the system will take an infinite time to reach stationarity and in principle will contain an infinite amount of energy. Note that this is not a catastrophe! In continuum problems, a catastrophe is when you get an infinite density of some kind. Here the work, transfer and dissipation rates are the densities of the problem, and they are perfectly well behaved.

At this stage, when I try to discuss the infinite Reynolds number limit, people tend to get uneasy and talk about possible singularities or discontinuities. I don’t really think that there is any cause for such hand-wringing. You have to decide first, which Navier-Stokes equation (NSE) you are using. There are two possibilities and they are identical; but we arrive at them by different routes.

If we arrive at the NSE by continuum mechanics, then in principle we can take the limit of zero viscosity without worry. After all, this is just a model of a real viscous fluid and, among other things, it is rigorously incompressible which a real fluid isn’t. We accept that in practice that it is the *flow* which is incompressible, not the fluid. So if the density variations are too small to detect, we can safely use the NSE.

If you come by the statistical physics route, then you must bound the smallest length scale (here the Kolmogorov dissipation length scale) such that it is orders of magnitude larger than inter-molecular distances. In practice, we may see the asymptotic behaviour associated with small viscosity arising long before there is any danger of breaching the continuum limit. For instance, if we look at the behaviour of the dimensionless dissipation rate as the Reynolds number is increased (see Fig. 1 of paper #6 in my list of recent papers) we are actually seeing the onset of the infinite Reynolds number limit. The accuracy of the determinations of $C_{\varepsilon,\infty}$ in this work is very decent, but if greater accuracy were required, then a bigger simulation would provide. Just like in boundary layer theory, it is all a matter of quite pragmatic considerations. I will give a more pedagogic discussion of this topic in a future post.