

A first look at Kolmogorov (1941)

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Around the turn of the new millennium, I attended the PhD oral of one of my own students for the last time as Internal Examiner. After that the regulations were changed; or perhaps it was frowned on for the supervisor to also be the Internal. Later still I stopped attending in any capacity: I think it became that the student had to invite their supervisor if they wanted them to attend. Is this an improvement on the previous system? Actually, my own PhD oral was conducted by David Leslie, who had previously been my second supervisor, and Sam Edwards who was my first supervisor! The three of us had had many discussions of my work in the past, so the atmosphere was informal and friendly. But I don't think the examination lacked rigour and I suppose it would have been difficult to find anyone else in the UK who could have acted as external examiner.

However, back to my own last stint as Internal. The candidate was a graduate with joint honours in maths and computer science. He was a very able young man and did good work, but he was not a physicist and never quite engaged with the physics. So when the External asked him if he could derive the Kolmogorov spectrum, he said 'No', then added pertly 'Can you?' Alas, the External was unable to do so. Fortunately the Internal was able to go to the blackboard and do the needful. The External was quite a well-known member of the turbulence community, so we will spare his blushes. Yet, it left me wondering how many turbulence researchers could sit down and derive the Kolmogorov energy spectrum, or equivalently the second-order structure function, without consulting a book?

For any such benighted souls, I will now offer a crib. Virtue should be its own reward, but in the process of putting this together, I think I have found the answer to something that had puzzled me. I will return to that at the end of this post.

For simplicity, let's work with the second-order structure function $S_2(r)$. This is what Kolmogorov did: the form for the energy spectrum came later. Glossing over the physical justification, we consider the question: how do we express $S_2(r)$ in terms of the dissipation rate ε and the distance between measuring points r , for some intermediate range of values of r ?

The first thing to notice is that S_2 has dimensions of velocity squared (or energy per unit mass: we won't keep repeating this) and that the dissipation is the rate of change of the energy with time. It follows that S_2 depends on the inverse of time squared whereas dissipation depends on the inverse of time cubed. Hence, the structure function must depend on the dissipation to the power of $2/3$. Or,

$$S_2(r) \sim \varepsilon^{2/3} r$$

This is the Kolmogorov result. Put in its most general form: if you seek to express the energy in terms of the dissipation, inertial transfer, eddy-decay rate, or any other rate of change, you must have a two-thirds power from the need to have consistency of the time dimension across both sides of the equation.

Now what happens when we tidy up the dimensions of length? On the right hand side of the equation, we now have the dimensions of length to the power of $4/3$. In order to make this consistent with S_2 on the left hand side, we must multiply by a length to the power of $2/3$. From Kolmogorov (1941), this length must be r , and if we put a constant C in front, we recover the well-known K41 result

$$S_2(r) = C r^{2/3} \varepsilon^{2/3}$$

If however, we think that it might also depend on another length, then we only have available some length characteristic of the size of the system, say L_{ext} . If we include this, then we must multiply the right hand side by $L_{\text{ext}}^p r^m$, where $p+m=2/3$. In other words, the power of r is no longer determined. This is, in effect, what Kolmogorov did in 1962, albeit by a more circuitous route. And, in the process he threw away his entire theory, which was based on the idea that the many steps of the Richardson cascade would lead to a universal result at small scales. In Kolmogorov (1962) that does not happen: the final result depends on the physical size of the system.

Let us now hark back to what had puzzled me. In a previous post I mentioned a contumacious referee. In fact this individual kept asserting that ' $r^{2/3}$ is not Kolmogorov'. We pressed him to explain but it was clear that he had found his excuse for rejecting the paper and wasn't prepared to be more helpful (or indeed scholarly). As our paper contained a discussion of the fact that the extended scale similarity technique gave the two-thirds law as an artifact in the dissipation range, it is possible that he was actually agreeing with us! However, taking his comment as a general statement, I would be inclined to agree with it. From the discussion we have given above, it should be clear that it is the dependence on the *dissipation rate* to the two-thirds power that is actually Kolmogorov. For anyone interested, the paper is Number 7 in the list of my recent papers given on this website.