Elliptic Flavoured PDEs - the Laplacian and Beyond

Erik Sätterqvist

Maxwell PG Colloquium

Three fundamental PDEs

- The wave equation (Hyperbolic)

$$\boxed{\Delta u(x,t) - \partial_{tt} u(x,t) = 0}$$

- The heat equation (Parabolic)

$$\boxed{\Delta u(x,t) - \partial_{t} u(x,t) = 0}$$

- The Laplace equation (Elliptic)

$$\boxed{\Delta u(x) = 0}$$

Here $(x,t) \in \mathbb{R}^{n+1}$ and $\Delta = \Delta_x = \partial_{11} + \cdots + \partial_{nn}$

The wave equation

$$\boxed{\Delta u(x,t) - \partial_{tt} u(x,t) = 0}$$

- Suppose

$$\partial_{tt}\big|_{t=t_0} u(x,t) = 0,$$

- Then

$$\Delta u(x,t_0) = 0.$$

The wave equation

The Dirichlet problem

$$\begin{cases} \Delta u(x,t) - \partial_{tt} u(x,t) = 0, \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x). \end{cases}$$

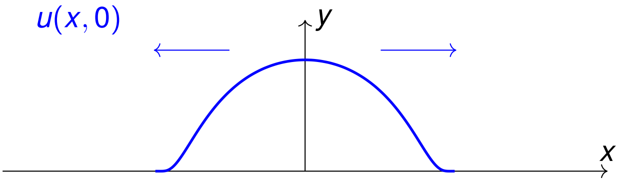
Solution given by D'lamberts formula

$$u(x,t) = \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x+t}^{x-t} g ds$$

Simple Dirichlet problem

Let

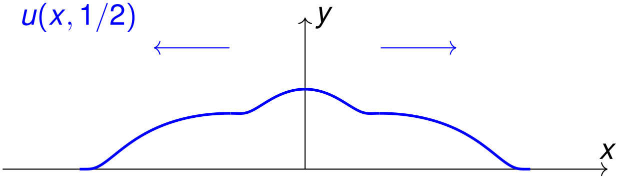
$$f(x) = e^{1/(x^2-1)}\chi_{(-1,1)}, \quad g(x) = 0$$



Simple Dirichlet problem

Let

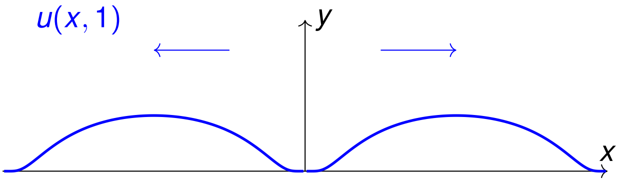
$$f(x) = e^{1/(x^2-1)}\chi_{(-1,1)}, \quad g(x) = 0$$



Simple Dirichlet problem

Let

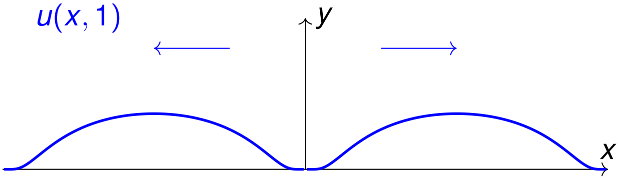
$$f(x) = e^{1/(x^2-1)}\chi_{(-1,1)}, \quad g(x) = 0$$



Simple Dirichlet problem

Let

$$f(x) = e^{1/(x^2-1)}\chi_{(-1,1)}, \quad g(x) = 0$$



- Finite propagation speed
- No hope of

$$\partial_{tt}\big|_{t=t_0} u(x,t) = 0,$$

The heat equation

$$\boxed{\Delta u(x,t) - \partial_t u(x,t) = 0}$$

- Suppose

$$\partial_t\big|_{t=t_0} u(x,t) = 0,$$

- Then

$$\Delta u(x,t_0) = 0.$$

The heat equation

The Dirichlet problem

$$\begin{cases} \Delta u(x,t) - \partial_t u(x,t) = 0, \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

The heat equation

The Dirichlet problem

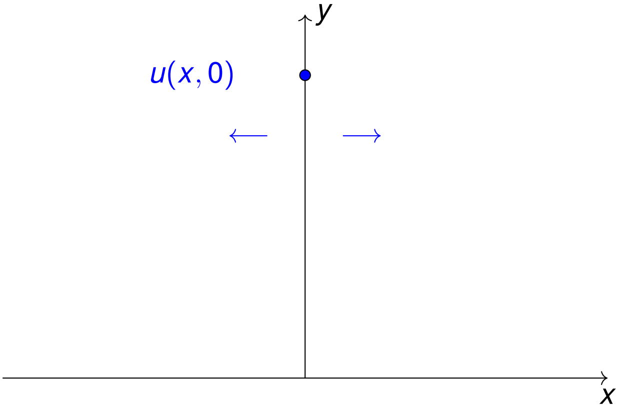
$$\begin{cases} \Delta u(x,t) - \partial_t u(x,t) = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

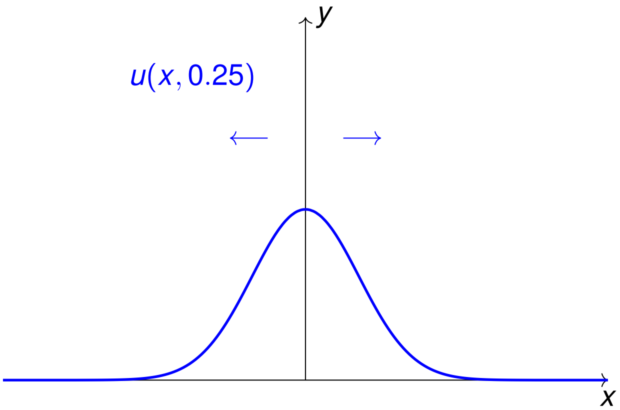
Let $\delta(x)$ be a dirac mass i.e. $\delta(x)[f] = f(x)$.

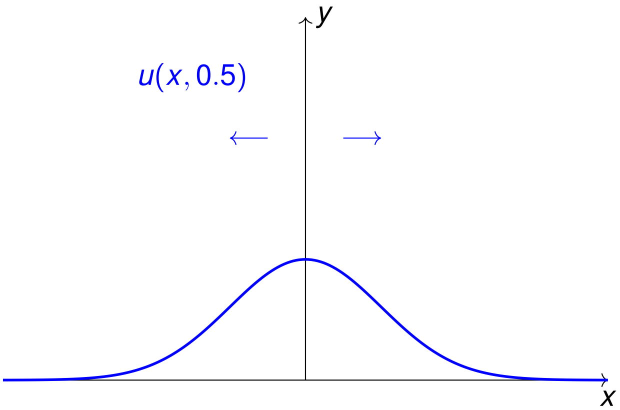
$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

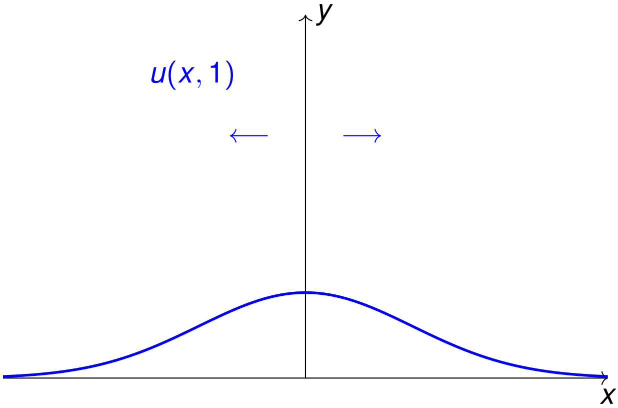
solves

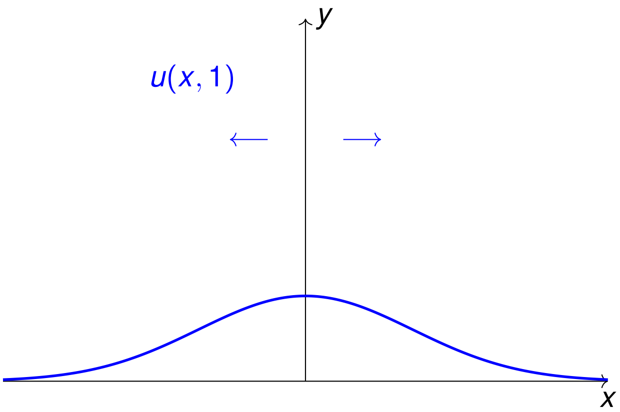
$$\begin{cases} \Delta u(x,t) - \partial_t u(x,t) = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \delta(x) \end{cases}$$


y
u(x, 0)
x


y
u(x, 0.25)
x


y
$u(x, 0.5)$
x


y
u(x, 1)
x



- Infinite propagation speed
- Uniform decay in t

$G(\cdot, t)$ Approximate identity

$$\begin{cases} \int_{\mathbb{R}} G(x,t)dx = 1 \\ G(x,t) = \frac{1}{\sqrt{t}} G(x/\sqrt{t}, 1) \end{cases}$$

So if $\phi \in C^0$ then

$$G(\cdot, t) * \phi \to \phi \text{ uniformly as } t \to 0,$$

where

$$[G(\cdot, t) * \phi](x) = \int_{\mathbb{R}} G(x-y,t)\phi(y)dy.$$

The wave equation

so

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{(y^2 - x^2)/4t} \phi(y)dy$$

solves

The Dirichlet problem

$$\begin{cases} \Delta u(x,t) - \partial_t u(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Still have

- Infinite propagation speed
- Uniform decay in t

The Heat equation

$$\boxed{\Delta u(x,t) - \partial_t u(x,t) = 0}$$

- It is reasonable to suppose

$$\partial_t\big|_{t=t_0} u(x,t) = 0,$$

- so then

$$\Delta u(x,t_0) = 0.$$

The Heat equation

$$\boxed{\Delta u(x,t) - \partial_t u(x,t) = 0}$$

- It is reasonable to suppose

$$\partial_t\big|_{t=t_0} u(x,t) = 0,$$

- so then

$$\Delta u(x,t_0) = 0.$$

- Can think of solution to $\boxed{\Delta u = 0}$ as steady heat flow

The Laplace equation

Let $n = 2$

$$\begin{aligned}\Delta u(x,y) &= \partial_{xx} u(x,y) + \partial_{yy} u(x,y) \\ &= \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h} \\ &\quad + \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h} + O(h^2)\end{aligned}$$

so $\boxed{\Delta u = 0}$ implies

$$u(x,y) = \frac{1}{4}\big(u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h)\big)$$

- $u(x,y)$ is average

The Laplace equation

$\boxed{\Delta u = 0}$ in domain $\Omega \subset \mathbb{R}^n$

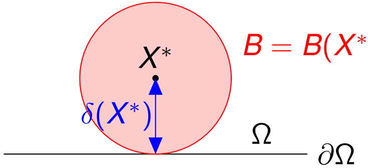
- $u(X)$ is spherical average

$$u(X) = \frac{1}{|\partial B(X,r)|} \int_{\partial B(X,r)} u d\sigma, \quad \bar{B}(X,r) \subset \Omega$$

Maximum principle

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$$

$$u(X^*) = \sup_{\Omega} u$$



$$B = B(X^*, \delta(X^*))$$

The Laplace equation

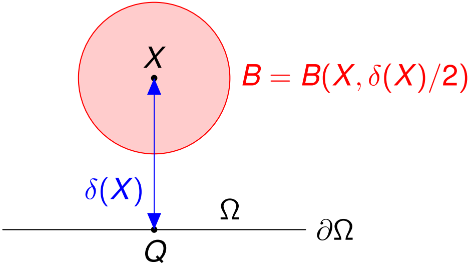
Let $\boxed{\Delta u = 0}$ and $u \geq 0$ in domain $\Omega \subset \mathbb{R}^n$

Harnack's inequality

$$\sup_B u \leq C \inf_B u$$

i.e.

$$u \approx \text{ constant on } B.$$



The Laplace equation

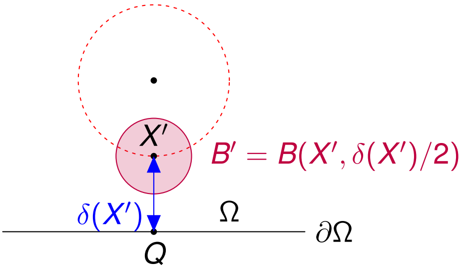
Let $\boxed{\Delta u = 0}$ and $u \geq 0$ in domain $\Omega \subset \mathbb{R}^n$

Harnack's inequality

$$\sup_B u \leq C \inf_B u$$

i.e.

$$u \approx \text{ constant on } B.$$



The Laplace equation

Let $\boxed{p > 1}$ and $f \in L^p(\partial\Omega)$ i.e. $\int_{\partial\Omega} |f|^p < \infty$

The Dirichlet problem

$$\begin{cases} \Delta u \overset{\text{w}}{=\!=} 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases} \qquad ([DP]_p)$$

- Where should we look for solutions? i.e. $u \in$? Bigger is better!

The Laplace equation

By the *fundamental theorem of calculus of variations*

$$\Delta u = 0 \text{ in } \Omega \iff \int_\Omega \Delta u \varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

IBP implies

$$\int_\Omega \Delta u \varphi = \int_\Omega \nabla u \cdot \nabla \varphi =: B_\Delta[u, \varphi]$$

Weak solution

$$\Delta u \overset{\mathrm{w}}{=\!=} 0 \text{ in } \Omega \iff B_\Delta[u, \varphi] = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

The Laplace equation

Weak solution

$$\Delta u \overset{\text{w}}{=} 0 \text{ in } \Omega \iff B_\Delta[u, \varphi] = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Only need **one** derivative so take

$$u \in W^{1,2}(\Omega) = \left\{ v : \int_\Omega |v|^2, \int_\Omega |\nabla v|^2 < \infty \right\}$$

It is known that

$$\Delta u \overset{\text{w}}{=} 0 \text{ in ball } B \implies \Delta u = 0 \text{ in } B$$

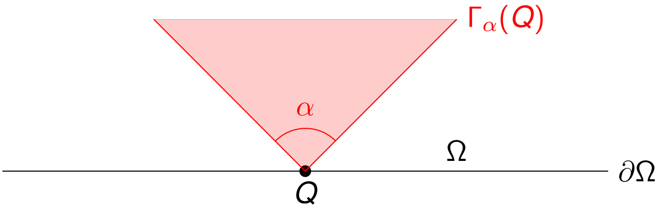
In fact $u \in C^\infty(B)$, *hypoellipticity*.

The Laplace equation

- $[DP]_p$ "easy" if $f \in C^0(\partial\Omega) \cap L^p(\partial\Omega)$
- Take $f_k \in C^0(\partial\Omega) \cap L^p(\partial\Omega)$ so that $f_k \to f$ in L^p
- If

$$\int_{\partial\Omega} |N[u_k]|^p \leq C \int_{\partial\Omega} |f_k|^p, \quad N[u](Q) = \sup_{\Gamma_\alpha(Q)} |u|$$

Then by *Maximum principle* $u_k \to u$ in $W^{1,2}_{\text{loc}}(\Omega)$ and u solves $[DP]_p$.



For $f \in C^0(\partial\Omega) \cap L^p(\partial\Omega)$ find $u \in W^{1,2}(\Omega)$ with

$$\begin{cases} \Delta u \overset{\text{w}}{=\!=} 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

and prove

$$\int_{\partial\Omega} |N[u]|^p \leq C \int_{\partial\Omega} |f|^p, \quad \forall f \in C^0(\partial\Omega) \cap L^p(\partial\Omega)$$

- In fact $u \in C^{0,\alpha}(\Omega)$ using the *Harnack inequality*
- Can we replace Δ with more general operators L?

Note $\Delta = \text{div}(I\nabla)$ & let $\boxed{L = \text{div}(A\nabla)}$ where $A = A(X) \in \mathbb{R}^{n\times n}$, $X \in \Omega$.
As before

$$\int_\Omega \text{div}(A\nabla u)\varphi = \int_\Omega A\nabla u \cdot \nabla\varphi =: B_L[u, \varphi].$$

Note

$$B_\Delta[u, u] = \int_\Omega |u|^2.$$

Need

$$\frac{1}{C}\int_\Omega |u|^2 \leq B_L[u, u] \leq C\int_\Omega |u|^2,$$

which follows if

$$\frac{1}{C}|\xi|^2 \leq A\xi \cdot \xi \leq C|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \ a.e. X \in \Omega.$$

The Laplace equation

Ellipticity

$L = \mathrm{div}(A\nabla)$ is *elliptic* if $\exists\lambda > 0$ s.t.

$$\lambda|\xi|^2 \leq A\xi\cdot\xi \leq \lambda^{-1}|\xi|^2, \quad \forall \xi\in\mathbb{R}^n,\ a.e. X\in\Omega.$$

- Thus $\boxed{A(X)\xi\cdot\xi = 1}$ defines an *ellipsoid* in $\xi\in\mathbb{R}^n$ with

$$\lambda \leq \text{the length of the semi-axi} \leq \lambda^{-1}$$

- Note

$$A\xi\cdot\xi = A^s\xi\cdot\xi, \quad A^s = \frac{1}{2}(A + A^T).$$

So $A^a := A - A^s$ free

Perturbation theory

To solve $[DP]_p$ for L we need it to be "close" to Δ

Dindos-Sätterqvist-Ulmer '22

- $\Omega \subset \mathbb{R}^n$ be bounded chord-arc domain.
- $L_0 = \mathrm{div}(A_0 \nabla)$ and $L_1 = \mathrm{div}(A_1 \nabla)$ elliptic with $\|A_i^a\|_{\mathrm{BMO}} \leq \Lambda$.

$$\exists r = r(n, \lambda, \Lambda) \in (1, \infty),\ \gamma = \gamma(n, \lambda, \Lambda) > 0,$$

so that if

$$\int_{T(\Delta)} \frac{\beta_r(Z)^2}{\delta(Z)} dZ \leq \gamma |\Delta|, \quad \beta_r(Z) := \left(\int_{B(Z, \delta(Z)/2)} |A_1 - A_0|^r \right)^{1/r}$$

then $[DP]_p$ for L_0 solvable implies $[DP]_p$ solvable for L_1,

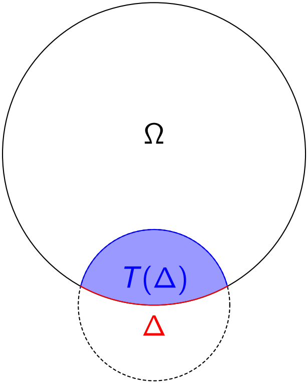
Perturbation theory

Can think of

$$\int_{T(\Delta)} \frac{\beta_r(Z)^2}{\delta(Z)} dZ \leq \gamma |\Delta|$$

as

$$\beta_r(Z) \sim \delta(Z)^{0+}$$



Perturbation theory

- Let $L_0 = \Delta$ and $L_1 = L = \mathrm{div}(A\nabla)$.
- Suppose

$$\int_{\partial\Omega} |N[u]|^p \leq C \int_{\partial\Omega} |f|^p, \quad \forall f \in C^0(\partial\Omega) \cap L^p(\partial\Omega)$$

so that $[DP]_p$ solvable for Δ.

- $[DP]_p$ solvable for L if

$$\beta_r(Z) = \left(\int_{B(Z,\delta(Z)/2)} |A - I|^r \right)^{1/r} \sim \delta(Z)^{0+}$$

i.e. $\boxed{|A(Z) - I| \sim \delta(Z)^{0+}}$

Thanks for listening!

Preprint: https://arxiv.org/abs/2207.12076

Email: erik.satterqvist@ed.ac.uk