The maths behind the music

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PG Colloquium, 4th March 2022

Where it all began

Ancient Greece

- Pythagorean scale
- Ratios of intervals

Galileo

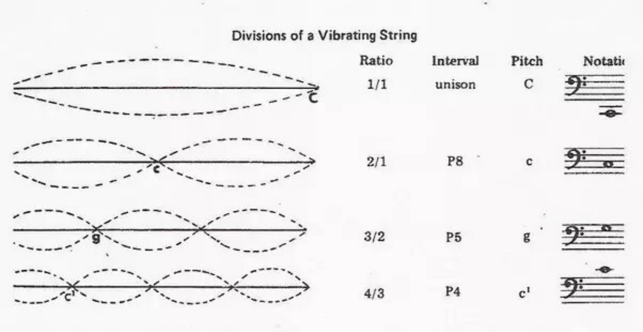
- Vibrations on strings

18th Century

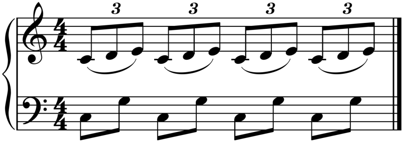
- Equal temperament

Modern connections

- Fractals
- Frieze patterns



Poly rhythms



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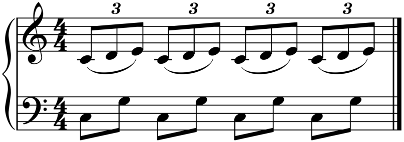
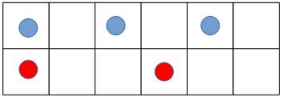
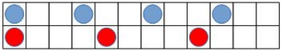


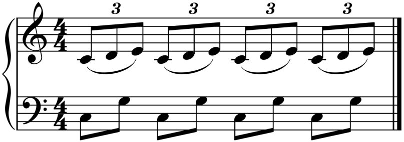
3 against 2 poly rhythm



4 against 3 poly rhythm



Poly rhythms



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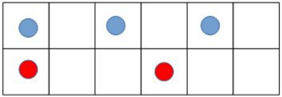

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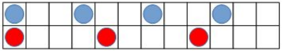
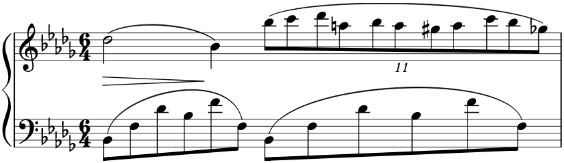
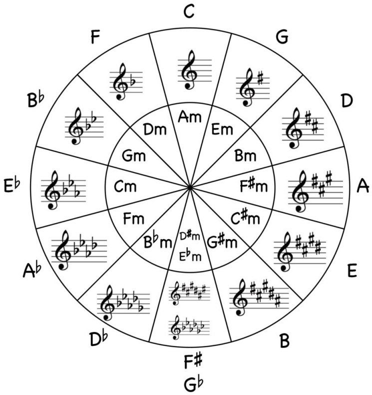




Circle of 5ths



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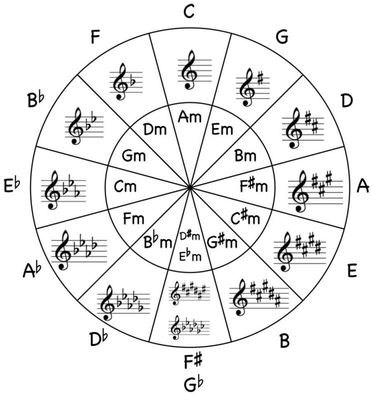

- Assign (C, C#, D, . . . , B) = (0,1,. . . , 11)
- Multiply by 7 = (0, 7, 14, . . . , 77)
- Apply mod 12 = (0, 7, 2, 9, . . . , 10, 5)
- Reassign letters = (C, G, D, . . . , B♭, F)

Algebra

Definition

A *group* is a set G equipped with a binary operation $*$ such that the following hold:

1. **Closure**: For all $x, y \in G$, $x * y \in G$
2. **Associativity**: For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$
3. **Identity**: There exists a unique element $e \in G$, called the *identity element*, such that for all $x \in G$, $x * e = e * x = x$
4. **Inverses**: For all $x \in G$, there exists $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$

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Example 1: $(\mathbb{Z}, +)$.
Example 2: Symmetries of regular polygons $\Rightarrow$ *Dihedral group*.
Example 3: $(\mathbb{Z}/n\mathbb{Z}, +)$, for example $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

Music connections

Note	C	C♯/$D\flat$	D	D♯/$E\flat$	E	F	F♯/$G\flat$	G	G♯/$A\flat$	A	A♯/$B\flat$	B
Label	0	1	2	3	4	5	6	7	8	9	10	11

Chromatic scale $\cong \mathbb{Z}/12\mathbb{Z}$

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3-note chords

Major = (Root, Major 3rd, Perfect 5th)

Minor = (Root, Minor 3rd, Perfect 5th)

Let $\mathcal{M}$ denote the set of all possible Major and Minor Chords.

$$\mathcal{M} = \{(x, x+3, x+7), (y, y+4, y+7) \mid x, y \in \mathbb{Z}/12\mathbb{Z}\}$$

Example: C Major chord = (C, E, G) $\Leftrightarrow (0, 4, 7) \in \mathcal{M}$

The $\mathcal{PLR}$ Group

Minor chords = $(a,b,c) \in \mathcal{M}$, Major chords = $(A,B,C) \in \mathcal{M}$.

Parallel	Leading note	Relative
Same letter, opposite parity	Semitone below	Opposite parity, same key signature
$\mathcal{P} : \mathcal{M} \to \mathcal{M}$	$\mathcal{L} : \mathcal{M} \to \mathcal{M}$	$\mathcal{R} : \mathcal{M} \to \mathcal{M}$
$(a,b,c) \mapsto (a,b+1,c)$	$(a,b,c) \mapsto (c+1,a,b)$	$(a,b,c) \mapsto (b,c,a-2)$
$(A,B,C) \mapsto (A,B-1,C)$	$(A,B,C) \mapsto (B,C,A-1)$	$(A,B,C) \mapsto (C+2,A,B)$

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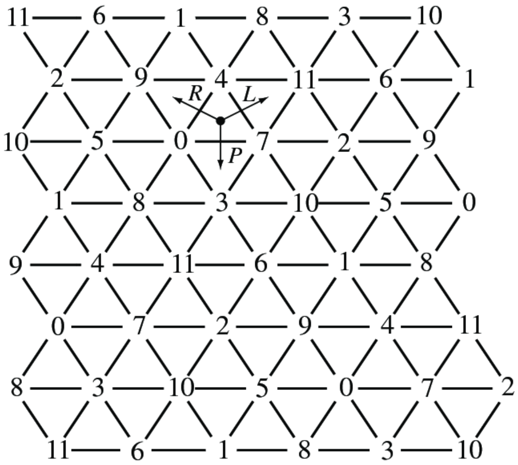
Examples:

C Maj = (C,E,G) = (0,4,7) $\xrightarrow{\mathcal{P}}$ (0,3,7) = (C,E♭,G) = C Min

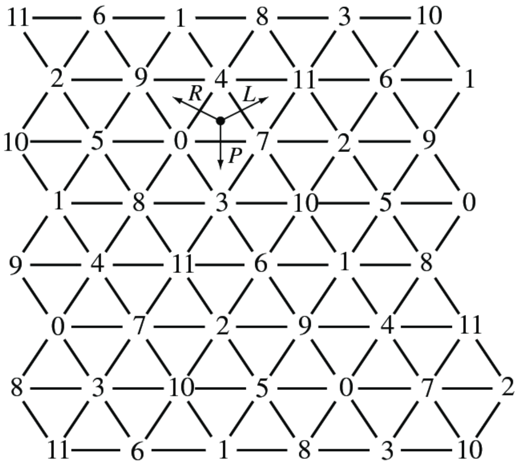
F Maj = (F,A,C) = (5,9,0) $\xrightarrow{\mathcal{L}}$ (4,9,0) = (A,C,E) = A Min

A Min = (A,C,E) = (9,0,4) $\xrightarrow{\mathcal{R}}$ (0,4,7) = (C,E,G) = C Maj



11 6 1 8 3 10
2 9 4 11 6 1
R L
10 5 0 7 2 9
P
1 8 3 10 5 0
9 4 11 6 1 8
0 7 2 9 4 11
8 3 10 5 0 7 2
11 6 1 8 3 10

The $\mathcal{PLR}$ Group



Theorem

The set $G = \langle \mathcal{P}, \mathcal{L}, \mathcal{R} \rangle$ form a group.

Transpositions and Inversions

Definition

Let $x = (a, b, c) \in \mathcal{M}$. A **transposition** is a function

$$\begin{aligned} T_n : \mathcal{M} &\to \mathcal{M} \\ x &\mapsto x + n \bmod 12 = (a + n, b + n, c + n) \end{aligned}$$

An **inversion** is a function

$$\begin{aligned} I_n : \mathcal{M} &\to \mathcal{M} \\ x &\mapsto -x + n \bmod 12 = (-a + n, -b + n, -c + n) \end{aligned}$$

Here $0 \leq n < 12$.

Transpositions and Inversions

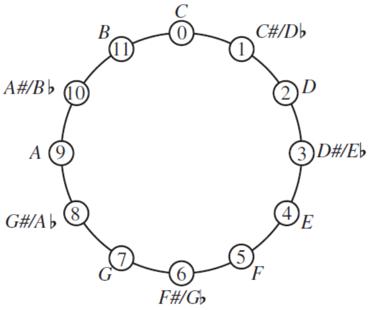


Figure 1: The musical clock.

Transpositions and Inversions

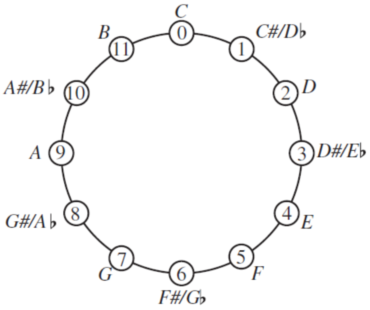


Figure 1: The musical clock.

Theorem

$$\langle T_1, I_0 \rangle \cong D_{12}$$

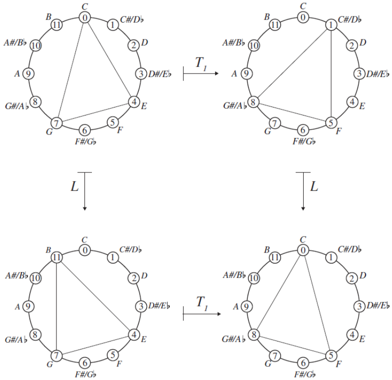
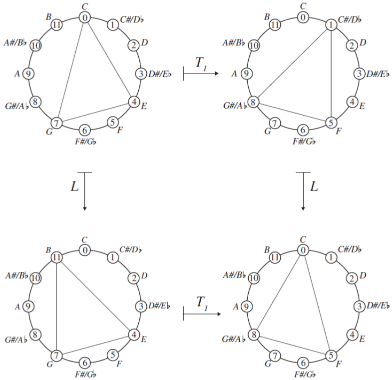


Figure 8: Illustration of commutativity of T_1 and L.

$R \longmapsto I_0$	$R \circ (LR)^4 \longmapsto I_8$	$R \circ (LR)^8 \longmapsto I_4$
$LR \longmapsto T_1$	$(LR)^5 \longmapsto T_5$	$(LR)^9 \longmapsto T_9$
$R \circ (LR) \longmapsto I_{11}$	$R \circ (LR)^5 \longmapsto I_7$	$R \circ (LR)^9 \longmapsto I_3$
$(LR)^2 \longmapsto T_2$	$(LR)^6 \longmapsto T_6$	$(LR)^{10} \longmapsto T_{10}$
$R \circ (LR)^2 \longmapsto I_{10}$	$R \circ (LR)^6 \longmapsto I_6$	$R \circ (LR)^{10} \longmapsto I_2$
$(LR)^3 \longmapsto T_3$	$(LR)^7 \longmapsto T_7$	$(LR)^{11} \longmapsto T_{11}$
$R \circ (LR)^3 \longmapsto I_9$	$R \circ (LR)^7 \longmapsto I_5$	$R \circ (LR)^{11} \longmapsto I_1$
$(LR)^4 \longmapsto T_4$	$(LR)^8 \longmapsto T_8$	$(LR)^0 \longmapsto T_0$

Table 4.3: The isomorphism $\phi : \mathrm{PLR} \mapsto \mathrm{TI}$.

Transpositions and Inversions



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$R \circ (LR) \longmapsto I_{11}$	$R \circ (LR)^5 \longmapsto I_7$	$R \circ (LR)^9 \longmapsto I_3$
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$(LR)^3 \longmapsto T_3$	$(LR)^7 \longmapsto T_7$	$(LR)^{11} \longmapsto T_{11}$
$R \circ (LR)^3 \longmapsto I_9$	$R \circ (LR)^7 \longmapsto I_5$	$R \circ (LR)^{11} \longmapsto I_1$
$(LR)^4 \longmapsto T_4$	$(LR)^8 \longmapsto T_8$	$(LR)^0 \longmapsto T_0$

Table 4.3: The isomorphism ϕ : PLR $\mapsto$ TI.

Theorem

$$\langle \mathcal{P}, \mathcal{L}, \mathcal{R} \rangle \cong \langle T_1, I_0 \rangle \cong D_{12}$$

References

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Thank you for listening!
Any questions?

"Music is the arithmetic of sounds as optics is the geometry of light"
Claude Debussy