# Uncovering Product Cannibalisation

## using Multivariate Hawkes Processes

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# Research Goal

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We aim to quantify product cannibalisation for existing products in an apparel wholesale data set.

> *Product cannibalisation refers to the decrease in sales of one product due to (the introduction of) a closely related product. [Copulsky, 1976]*

The current focus is on inference to detect and understand product cannibalisation for products that already have a sales history.

Three questions you will be able to answer after this talk:

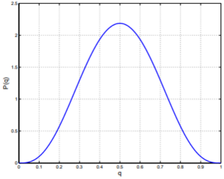
1. What is a Hawkes Process?
2. Why do we reparametrise the Hawkes Process?
3. How can a Hawkes Process be used to describe product cannibalisation?

# Prerequisites

# Bayesian Statistics: Parameter as a Matter of Interest

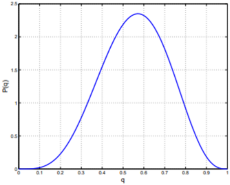
Frequentist Statistics: t-Test, F-Test, p-values...

Bayesian Statistics: parameters now have a distribution!
Prior → Data → Posterior



**(a)** Prior

**(b)** Data



**(c)** Posterior

Data: event times (plus additional covariates)
Let $N(t)$ be the number of observed events from $0$ to $t$.
Homogeneous ($\lambda$ constant) Poisson point process:

$$\mathbb{P}[N(t) = N] = \frac{(\lambda t)^N}{N!} e^{-\lambda t} \tag{1}$$

$$\mathbb{E}[N(t)] = \lambda t \tag{2}$$

Inhomogeneous ($\lambda(t)$ variable) Poisson point process:

$$p(t_1 \ldots t_N) = \prod_{i=1}^{N} \lambda\left(t_i\right) e^{-\int_0^\infty \lambda(z)\, dz} \tag{3}$$

$$\mathbb{E}[N(t)] = \int_0^t \lambda(z)\, dz \tag{4}$$

[Daley and Vere-Jones, 2003]

# Univariate Hawkes Process

Hawkes processes [Hawkes, 1971] are a class of point processes that are used to model event data when the events can occur in clusters or bursts. They are defined on the interval $[0, T]$ with conditional intensity function:

$$\lambda(t|\theta) = \lambda_0(t) + \sum_{i:t>t_i} \nu(t - t_i) \tag{5}$$

Here, $\lambda_0(t)$ can capture seasonality and underlying trends, but we use $\lambda_0(t) = \mu$, and $\nu(t - t_i) = K\,\beta e^{-\beta(t-t_i)}$ the self-excitement.

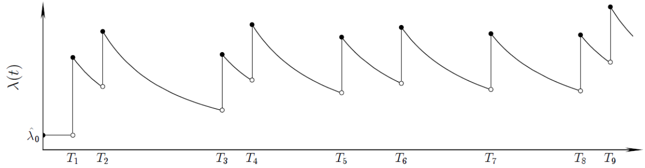



**Figure 2:** Intensity function with self exciting kernel [Rizoiu et al., 2017]

# Model

Assume that there are $M$ dimensions with data $Y_1 = (t_{1\,1} \ldots t_{1\,N_1}) \ldots Y_M = (t_{M\,1} \ldots t_{M\,N_M})$. At time $t$ the intensity in dimension $i$ is defined as the sum of the background rate $\mu_i(t)$ and contributions from all dimensions:

$$\lambda_i(t) = \left[\mu_i(t) + \sum_{j=1}^{M} \sum_{l:t>t_{j\,l}} K_{ji}\, g_{ji}(t - t_{j\,l})\right]_+ \tag{6}$$

We assume the following form for the influence for all $i, j$: $g_{ij}(x) > 0$ for $x > 0$ and $\int_0^\infty g_{ij}(x)\, dx = 1$. Here, each $K_{ij} < 1$, and we write them as matrix $\mathbf{K} = \{K_{ij}\}$ where $i, j = 1 \ldots M$.

For three dimensions $(A, B, C)$ we can estimate the parameters $\mathbf{K}$ in a variety of ways, e.g. Maximum Likelihood, Bayesian Posterior Mean. The resulting estimates can be visualised in a network-like structure.

$$\mathbf{K} = \begin{bmatrix} +0.5 & +0.75 & . \\ . & . & -0.25 \\ . & . & +0.25 \end{bmatrix}$$

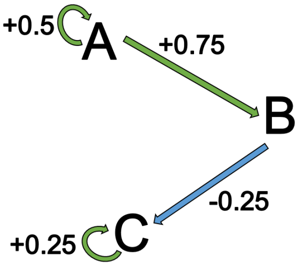



**Figure 3:** Graph example of the parameters $\mathbf{K}$ for three products

The entry $K_{ij}$ describe how many direct events an event in $i$ 'triggers/inhibits' in $j$. However, this does not take the global view into account.

Therefore, it could be of interest to look at the marginal effects $\mathbf{K}^*$ as they summarise the influence across the whole network.

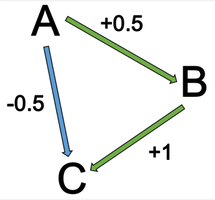



**Figure 4:** Local influences

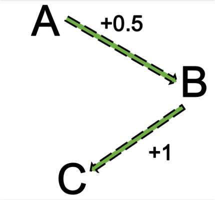



**Figure 5:** Global influences

# Mathematically Convenient $\mathbf{K}^*$

The number of events in a Hawkes Process can 'explode' when $\mathbf{K}$ gets too large.

When we add dimensions, the entries of $\mathbf{K}$ need to become smaller to retain stability. In contrast, $\mathbf{K}^*$ are dimension-independent (because they sum over all 'paths').

We therefore reparametrise the model in terms of $\mathbf{K}^*$, the marginal influences. At time $t$ the intensity in dimension $i$ is defined as the sum of the background rate $\mu_i(t)$ and contributions from all dimensions:

$$\lambda_i(t) = \left[ \mu_i(t) + \sum_{j=1}^{M} \sum_{l:t>t_{j\,l}} \{f(\mathbf{K}^*)\}_{ji} \, g_{ji}(t - t_{j\,l}) \right]_+ \tag{7}$$

where $f(\mathbf{X}) = I - (\mathbf{X} - I)^{-1}$.

To implement this model we needed to overcome a few challenges:

- Ensuring a non-negative intensity $\rightarrow$ link function
- Integrating the intensity $\rightarrow$ numerical approximation
- Checking for stability $\rightarrow$ new criterion

All methodological details can be found in the mathematical draft on ArXiv [Deutsch and Ross, 2022].

# DATA

We use a multivariate Hawkes Process where each product represents one dimension. This allows us to estimate the 'influence' $K_{ij}$ from an event (sale) of one product $i$ onto each product $j = 1 \dots M$.

A positive influence $K_{ij} > 0$ is called excitation, a negative influence $K_{ij} < 0$ is referred to as inhibition. The latter is interpreted as product cannibalisation.

We fit the following multivariate Hawkes process in a Bayesian manner. The intensity in dimension $i$ is

$$\lambda_i(t) = \left[ \mu_i(t) + \sum_{j=1}^{M} \sum_{l:t>t_{j\,l}} \{f(\mathbf{K}^*)\}_{ji}\; g_{ji}(t - t_{j\,l}) \right]_+$$

For $\mu_i(t)$ we choose a step function with pre-defined change points where each product has an on-season and off-season background rate. Their priors are independent:

$$\begin{aligned} \mu_{i,\,\text{on}} &\sim \mathcal{U}(0, 10) && \text{for } i = 1 \ldots M \\ \mu_{i,\,\text{off}} &\sim \mathcal{U}(0, 10) && \text{for } i = 1 \ldots M \end{aligned}$$

For the influence kernels we utilise the popular exponential kernel $g_{ij}(x) = \beta_{ij}\, exp\,(-\beta_{ij}\, x)$. Here, we assume that all $\beta_{ii} = \beta_{\text{diag}}$ and $\beta_{ij} = \beta_{\text{off}}$ when $i \neq j$.

$$\beta_{\text{diag}} \sim \mathcal{U}(0, 3)$$
$$\beta_{\text{off}} \sim \mathcal{U}(0, 3)$$

In line with our previous arguments we place priors on the entries of $\mathbf{K}^*$. The estimation (using Stan) is carried out both using Normal priors

$$K^*_{ij} \sim \mathcal{N}(0, 1) \qquad \text{for } i, j = 1 \ldots M$$

and sparsity-inducing horseshoe priors

$$\xi_{ij} \sim \text{Cauchy}(0, 1)$$
$$K^*_{ij} \sim \mathcal{N}(0, \xi_{ij}) \qquad \text{for } i, j = 1 \ldots M$$

| | **Product 1** | **Product 2** | **Product 3** | **Product 4** |
|---|---|---|---|---|
| Main Colour | black | black | white | white |
| Branding | white | minimal | minimal | green |
| Label | none | known | known | known |

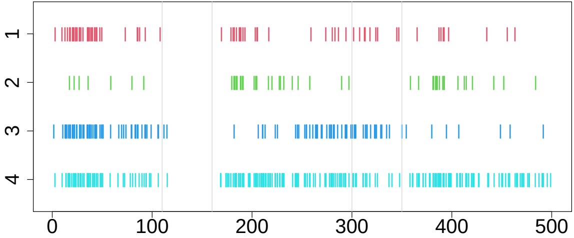



**Figure 6:** Observations

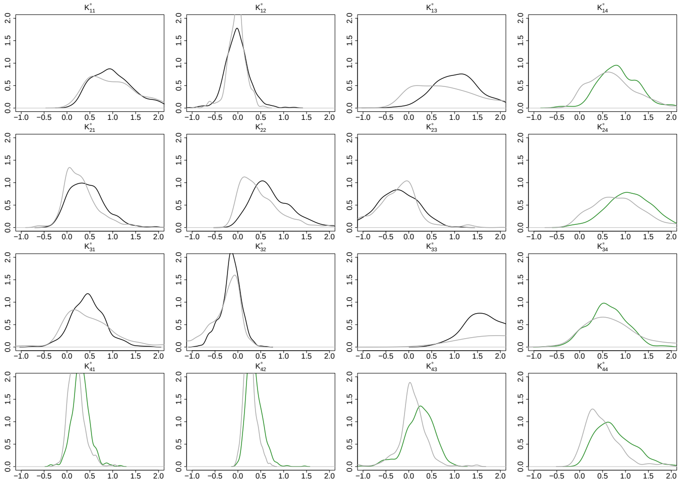



**Figure 7:** Posterior density estimates of $\mathbf{K}^*$ (excitation/inhibition parameter), using independent Normal (black) and horseshoe (grey) priors.

- Most products do not cannibalise each other as the posterior mass is mostly above zero.
- However, Product 2 and Product 3 display product cannibalisation in both directions ($K^*_{23} < 0$ and $K^*_{32} < 0$). Product 2 and 3 come from the same label. Maybe therefore wholesalers make the decision to only order one of the two due to their similar branding and label.
- Product 4 has the most sale events. It differs from the other products as it is the only one featuring a colour (green branding on the heel). This very popular style seems unaffected by product cannibalisation from other products.

# SUMMARY

Three questions you can now answer:

1. What is a Hawkes Process?
2. Why do we reparametrise the Hawkes Process?
3. How can a Hawkes Process be used to describe product cannibalisation?

**Methodological Advances**
to make the implementation of a multivariate Hawkes Process with inhibition easier.

**Formalisation of Product Cannibalisation**
as a mathematical concept that can be estimated, monitored, and predicted.

[Deutsch and Ross, 2022]
arxiv.org/abs/2201.05009

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