Knot energies of tangent-point-type and their generalization to higher dimensions

PG Colloquium Edinburgh

Axel Wings
wings@instmath.rwth-aachen.de
Joint work with Heiko von der Mosel and Henrik Schumacher

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Overview

- About me
- What is a knot?
  - How do we describe knots?
- Knot energies of tangent-point-type
  - The tangent-point radius
  - Definition
  - Some properties
- Generalization for surfaces and higher dimensions
  - Definition for sets
  - Definition via embeddings
  - Numerical Experiments

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About Me

Name:	Axel Wings
Position:	Doctoral researcher & teaching assistant (since Oct. 2020)
Topic:	Tangent-point energy in arbitrary dimensions
Advisor:	Heiko von der Mosel
University:	RWTH Aachen University
Insitute:	Institute for Mathematics
Associated to:	Research training group: EDDy
	Energy, Entropy, and Dissipative Dynamics
Email:	wings@instmath.rwth-aachen.de
Websites:	www.instmath.rwth-aachen.de
	www.eddy.rwth-aachen.de (check for open PhD positions)

What is a knot?

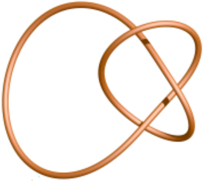
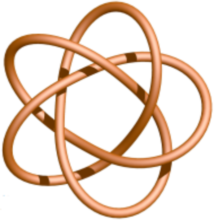
Teaser

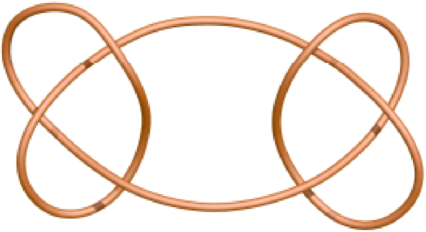
Real life example

simulation and graphics from [YSC21].

What is a knot?

Informal: A knot $\Sigma \subset \mathbb{R}^3$ is a closed line without self-intersection.



Graphics taken from [KSSvdM21]

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What is a knot?

How do we describe knots?

A introduction to knots can be found in [BZH13].

Definition: Knots as embeddings

A knot is an topological embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$, i. e. $\gamma : \mathbb{S}^1 \to \gamma(\mathbb{S}^1)$ is a homeomorphism.

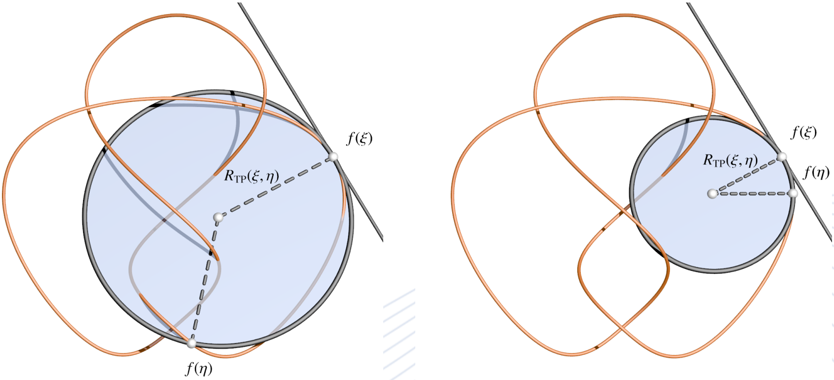
Definition: Domain of knot energies of tangent-point-type

The domain of tangent-point-type energies is given by
$\mathcal{C} := \{\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \mid |\gamma'| > 0 \text{ a. e.}\}$.

Remark: We intentionally allow self-intersections at this point. But if we restrict to injective curves, we get topological embeddings.

Knot energies of tangent-point-type

The tangent-point radius



Graphics created by Henrik Schumacher

Knot energies of tangent-point-type

The tangent-point radius

Consider the domain $\mathcal{C} = \{\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \mid |\gamma'| > 0 \text{ a.e.}\}$.

Definition: Tangent-point radius (cf. [GM99])

Let $\gamma \in \mathcal{C}$. Further let $u \in \mathbb{R}/\mathbb{Z}$ and $w \in [-\frac{1}{2}, \frac{1}{2}]$.
If $\gamma'(u)$ exists, $\gamma'(u) \neq 0$ and $\gamma(u+w) \notin \gamma(u) + \mathbb{R}\gamma'(u)$, then the **tangent-point radius** $r_{\text{tp}}[\gamma](u, u+w)$ is defined as the radius of the unique circle which has the these properties

1. $\gamma(u)$ and $\gamma(u+w)$ lie on the circle.
2. The circle is tangential to $\gamma'(u)$ in $\gamma(u)$.

Otherwise, it is set to infinity.

Remark: For fixed $u \in \mathbb{R}/\mathbb{Z}$ and $w \in [-\frac{1}{2}, \frac{1}{2}]$, the radii $r_{\text{tp}}[\gamma](u, u+w)$ and $r_{\text{tp}}[\gamma](u+w, u)$ are possibly different.

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Knot energies of tangent-point-type

Definition: Tangent-point energy

Definition: tangent-point energy TP_p (cf. [GM99])

Let $p > 0$. The **tangent-point energy** TP_p is defined by

$$\mathrm{TP}_p : \mathcal{C} \to \mathbb{R}_{\geq 0} \cup \{\infty\},$$

$$\gamma \mapsto 2^{-p} \cdot \int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{r_{\mathrm{tp}}[\gamma](u, u+w)} \right)^p |\gamma'(u)| \cdot |\gamma'(u+w)| \, dw \, du.$$

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Knot energies of tangent-point-type

The tangent-point radius

Lemma: Expressions for the tangent-point radius

Let $\gamma \in \mathcal{C}$. Further, let $u \in \mathbb{R}/\mathbb{Z}$ and $w \in (-\frac{1}{2}, \frac{1}{2})$. Then the following values are equal:

1. $r_{\mathrm{tp}}[\gamma](u, u+w)$
2. $\frac{|\gamma(u+w)-\gamma(u)|^2}{2|P^{\perp}_{\gamma'(u)}(\gamma(u+w)-\gamma(u))|}$
3. $\frac{|\gamma'(u)|\cdot|\gamma(u+w)-\gamma(u)|^2}{2|\gamma'(u)\times(\gamma(u+w)-\gamma(u))}$
4. $\frac{|\gamma(u+w)-\gamma(u)|^2}{2\,\mathrm{dist}(\gamma(u+w),\gamma(u)+\mathbb{R}\gamma'(u))}$

Important observation: We can split in numerator and denominator.

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Knot energies of tangent-point-type

Definition: Knot energies of tangent-point-type

Definition: knot energies of tangent-point-type TP_p^q (cf. [BR15a])

Let $p, q > 0$. The **knot energy of tangent-point-type** $\mathrm{TP}^{(p,q)}$ is defined by

$$\mathrm{TP}^{(p,q)} : \mathcal{C} \to \mathbb{R}_{\geq 0} \cup \{\infty\},$$
$$\gamma \mapsto \int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|P^{\perp}_{\gamma'(u)}(\gamma(u+w) - \gamma(u))|^q}{|\gamma(u+w) - \gamma(u)|^p} |\gamma'(u)| \cdot |\gamma'(u+w)| dw\, du.$$

Remark: $\mathrm{TP}_p = \mathrm{TP}^{(p,2p)}$

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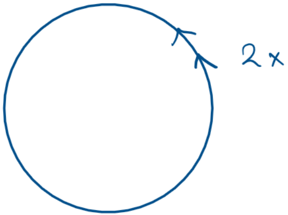
Knot energies of tangent-point-type

Some properties

The knot is homeomorphic to $\mathbb{S}^1$, if the energy is finite.

Theorem (cf. [SvdM12])

Let $p > 2$. If $\Gamma \in \mathcal{C}$ with $|\Gamma'| = 1$ a.e.(i.e. parameterized by arc-length), $\mathrm{TP}_p(\Gamma) < \infty$, and $\mathcal{H}^1(\Gamma(\mathbb{R}/\mathbb{Z})) = \mathfrak{L}^1(\Gamma)$ (i.e. the Hausdorff measure of the image equals the length of the curve), then $\Gamma(\mathbb{R}/\mathbb{Z})$ is homeomorphic to $\mathbb{S}^1$ and $\Gamma|_{[0,1)}$ is injective.



Knot energies of tangent-point-type

Some properties

Regularizing effects: The energy space is a fractional Sobolev space.

Theorem (cf. [Bla13])

Let $p > 2$ and $\Gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\Gamma'| \equiv 1$ and $\Gamma|_{[0,1)}$ injective. Then the following equivalence holds

$$\mathrm{TP}_p(\Gamma) < \infty \Leftrightarrow \Gamma \in W^{2-\frac{1}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3).$$

Remark

By definition of the fractional Sobolev space $\Gamma \in W^{2-\frac{1}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, iff

$$\Gamma \in W^{1,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$$

$$\text{and } \left(\int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\Gamma'(u+w) - \Gamma'(u)|^p}{|w|^p} \right)^{\frac{1}{p}} < \infty$$

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Knot energies of tangent-point-type

Some properties

Theorem: Continuous differentiability, cf. [Win18]

Let $q \in (1, \infty)$ and $p \in (q+2, 2q+1)$. Then, is the mapping

$$\delta \mathrm{TP}^{(p,q)} : W_{ir}^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \to \left(W^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \right)^*, \quad \gamma \mapsto \delta \mathrm{TP}^{(p,q)}(\gamma, \bullet)$$

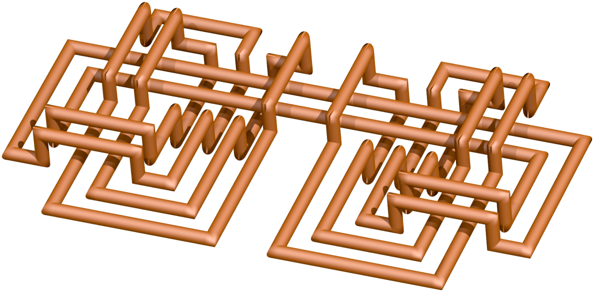
continuous, i.e. $\mathrm{TP}^{(p,q)}$ is Fréchet differentiable on $W_{ir}^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.

The first variation is calculated in [BR15b].

Knot-energies of tangent-point-type

Numerical simulation

Let's detangle this knot with a tangent-point energy and a gradient flow.



simulation and graphics from [YSC21]

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Generalization for surfaces and higher dimensions

Definition for Sets

By $G(n, m)$ we denote the Grassmannian consisting of all m-dimensional subspaces of $\mathbb{R}^n$.

Definition, cf. [BGMM03]

Let $\Sigma \subset \mathbb{R}^n$, $P : \Sigma \to G(n, m)$ and $p, q > 0$.
The tangent-point energy of Σ with parameters p and q is given by

$$\mathrm{TP}^{(p,q)}(\Sigma, P) := \int_\Sigma \int_\Sigma \frac{\operatorname{dist}(b, a + P(a))^q}{|b - a|^p} \, d\mathcal{H}^m(b) \, d\mathcal{H}^m(a).$$

Moreover, it is set

$$\mathrm{r}_{\mathrm{tp}}^{(p,q)}(P, a, b) := \frac{|b - a|^p}{\operatorname{dist}(b, a + P(a))^q}.$$

studied by [vdMS13] and [Kä21]

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Generalization for surfaces and higher dimensions

Definition via embeddings

Definition

Let $p, q > 0$. Further, let M be a closed, m-dimensional manifold. For a sufficiently smooth embedding $f : M \to \mathbb{R}^n$ the **generalized tangent-point energy** is given by

$$\mathcal{E}_p^q(f) := \iint_{M^2} E_p^q(f)(x,y)\omega_f(x)\omega_f(y),$$

where the integrand is given by

$$E_p^q(f)(x,y) := \frac{|(1-\mathcal{D}_f f)(x)(f(y)-f(x))|^q}{|f(y)-f(x)|^p}$$

Here, $\mathcal{D}_f f(x) := df|_x(df|_x)^\dagger \in \mathrm{Hom}(\mathbb{R}^n;\mathbb{R}^n)$ is the projector on $df|_x(T_xM)$.

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Generalization for surfaces and higher dimensions

Comparison of the definitions

Both definitions agree.

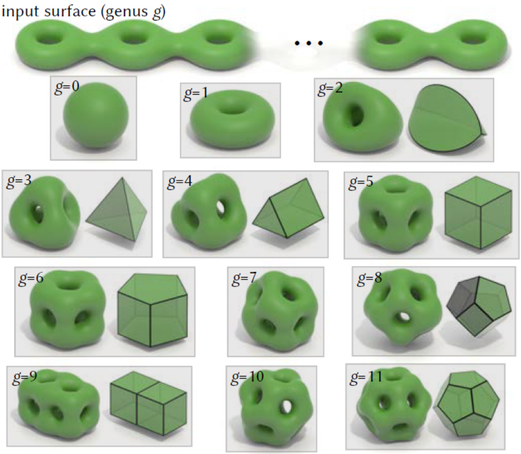
Theorem

Let M be a compact, m-dimensional manifold and $f : M \to \mathbb{R}^n$ a sufficiently smooth embedding. Further, let $P : f(M) \to G(n,m)$ be defined by $P(f(x)) = \mathrm{range}(df|_x)$ for all $f(x) \in f(M)$. Then it holds

$$\mathrm{TP}^{(p,q)}(f(M), P) = \mathcal{E}_p^q(f).$$

Generalization to surfaces

Numerical experiments



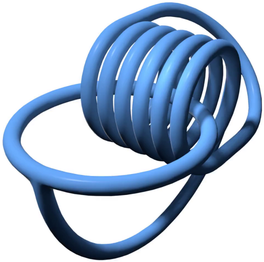
Taken from [YBSC21].

Recommendation: The YouTube Channel Two minute papers discussed the paper [YSC21] and [YBSC21]: https://youtu.be/M0RuBETA2f4

Generalization to surfaces

Numerical experiments

Let's detangle this surface with a tangent-point energy and a gradient flow.



simulation and graphics from [YBSC21]

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Generalization to surfaces

Perspective and conclusion

The fine properties of the 1D tangent-point energies and the numerical simulations motivate further studies of the generalization to higher dimensions.

Discussion

Thanks for the invitation and your attention.

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