The Dirichlet Problem for 2nd order linear elliptic PDEs

When can we solve them?

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12/11/21

Maxwell PG Colloquium

The Dirichlet Problem for elliptic PDEs

- $\Omega \subset \mathbb{R}^n$ domain
- Elliptic real matrix, i.e. $A(x) \in \mathbb{R}^{n \times n}$ with

$$\lambda|\zeta|^2 \leq \zeta^T A(x)\zeta \leq \frac{1}{\lambda}|\zeta|^2 \qquad \text{for all } x \in \Omega.$$

- Dirichlet Problem: Let $f \in C(\partial\Omega)$. Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{aligned} \mathcal{L}u = \mathrm{div}(A\nabla u) = 0 \qquad & \text{in } \Omega, \\ u = f \qquad & \text{on } \partial\Omega. \end{aligned}$$

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Example:

$$\begin{aligned} \operatorname{div}(\mathbb{I}\nabla u) = \Delta u = 0 \qquad & \text{in } \mathbb{B}, \\ u = f \qquad & \text{on } \partial\mathbb{B}. \end{aligned}$$

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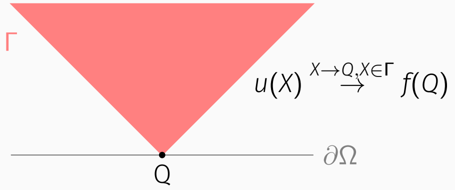
Weak solvability:

$$\int_\Omega A\nabla u \cdot \nabla\varphi dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)$$



Boundary data:

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We say the <u>L^p Dirichlet Problem for $\mathcal{L}$ is solvable</u>, if for all $f \in L^p(\partial\Omega) \cap C(\partial\Omega)$ the solution $u \in W^{1,2}(\Omega)$ of

$$\mathcal{L}u = \operatorname{div}(A\nabla u) = 0 \qquad \text{in } \Omega,$$
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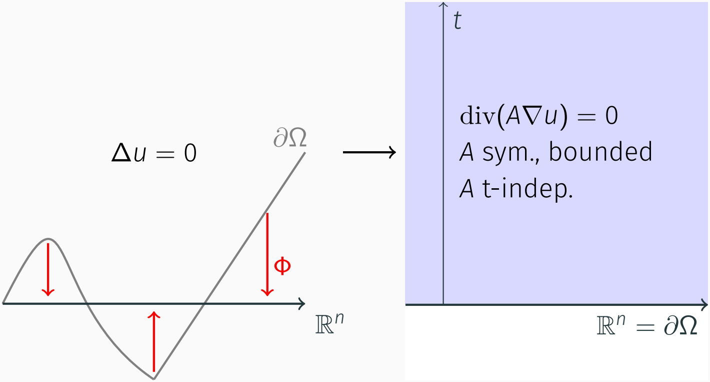
satisfies

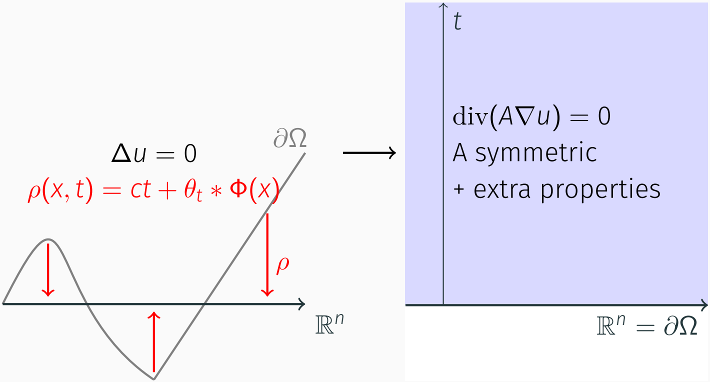
$$\|\tilde{N}(u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}.$$

From the Laplacian to the t-independence condition



It turns out that the L^p DP is solvable assuming t-independence.

From the Laplacian to the Carleson condition



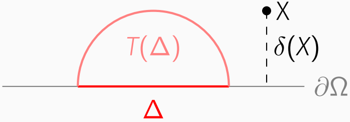
A has the properties

(i) $|\nabla A(x,t)| \leq \frac{C}{t}$,

(ii) $|\nabla A(x,t)|^2 t dx dt$ is a Carleson measure.

Solvability of L^p Dirichlet Problem for $\mathcal{L}$: Carleson condition

- $\delta(Z) := \mathrm{dist}(Z, \partial\Omega), \Delta(Q, r) := B(Q, r) \cap \partial\Omega, T(\Delta) := B(Q, r) \cap \Omega$



- A measure μ on $\mathbb{R}^n$ is called Carleson measure if there exists $C < \infty$ s.t

$$\mu(T(\Delta)) = \int_{T(\Delta)} d\mu(X) \leq C\sigma(\Delta) \qquad \text{for all } \Delta \subset \partial\Omega.$$

Smallest C with this property is called Carleson norm $\|\mu\|_{\mathcal{C}}$.

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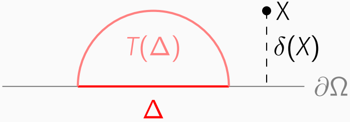

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- Set

$$d\mu(X) := \sup_{B(X,\delta(X)/2)} |\nabla A|^2 \delta(X) dX.$$

  Theorem: If $\|\mu\|_{\mathcal{C}} < \infty$ then $\mathcal{L}$ is L^p solvable for some p.
  If $\|\mu\|_{\mathcal{C}}$ is sufficiently small then $\mathcal{L}$ is L^2 solvable.

Pertubation

- Given:

$$\mathcal{L}_0 = \mathrm{div}(A_0 \nabla \cdot), \qquad \mathcal{L}_1 = \mathrm{div}(A_1 \nabla \cdot)$$

- Goal:

$$\mathcal{L}_0 \text{ is } L^p \text{ solvable} + \text{"}A_0 \text{ and } A_1 \text{ are close to each other"}$$
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$$\delta(Z) := \operatorname{dist}(Z, \partial\Omega), \qquad \alpha(Z) := \sup_{Y \in B(Z, \delta(Z)/2)} |A_0(Y) - A_1(Y)|$$

Theorem: Pertubation result with Carleson condition

If $\mathcal{L}_0$ is L^p solvable and

$$\| \frac{\alpha^2(X)}{\delta(X)} \|_{\mathcal{C}} \leq \gamma, \qquad \text{i.e.} \int_{T(\Delta)} \frac{\alpha(Z)^2}{\delta(Z)} dZ \leq \gamma \sigma(\Delta)$$

sufficiently small then $\mathcal{L}_1$ is L^p solvable.

Conclusion and further questions?

There are three (common) solvability criteria:

- t-independence
- Carleson measure condition
- Pertubation

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Question 1: What happens if A is non-symmetric?

Question 2: What happens if A is not bounded?

Question 3: Are there other conditions on pertubation between A_0 and A_1 so that solvability transfers from A_0 to A_1?

Question 4: Could there be a different type of perturbation which allows for non-trivial differences of the matrices at the boundary?

Question 5: What happens if A is a complex matrix?

Thank you for your attention! Questions?