Monte Carlo Elliptic Integrals

Circumference of an ellipse? Can’t do it mate.

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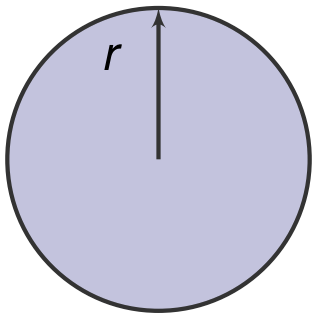
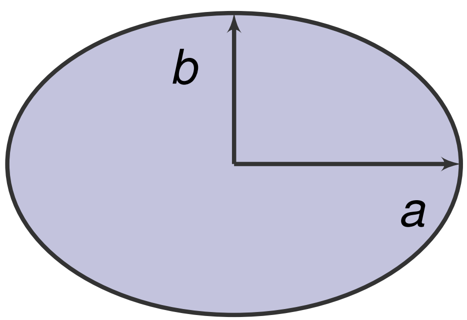
Background



Circle Area = πr^2
Circumference = $2\pi r$



Ellipse Area = πab
Circumference = ???

The Problem

We remember that a curve's arc length between $x = x_1$ and $x = x_2$ is:

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x.$$

For a circle

$$x^2 + y^2 = r^2,$$

and using $x_1 = 0$, $x_2 = r$ to work out a quarter gives the full circumference as

$$s = 4 \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, \mathrm{d}x$$

which after a bunch of maths spits out

$$s = 2\pi r.$$

For an ellipse

$$(x/a)^2 + (y/b)^2 = 1,$$

and using $x_1 = 0$, $x_2 = a$ again to work out a quarter gives the full circumference as

$$s = 4 \int_0^a \sqrt{1 + \frac{b^2 x^2}{a^4 - a^2 x^2}}\, \mathrm{d}x$$

which after a bunch of maths spits out

$$s = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, \mathrm{d}\theta.$$

Rough Approximations

The greats had a go:

1. Geometric Average

$$s = 2\pi\sqrt{ab}$$

(*Johannes Kepler, 1609*)

2. Arithmetic Average

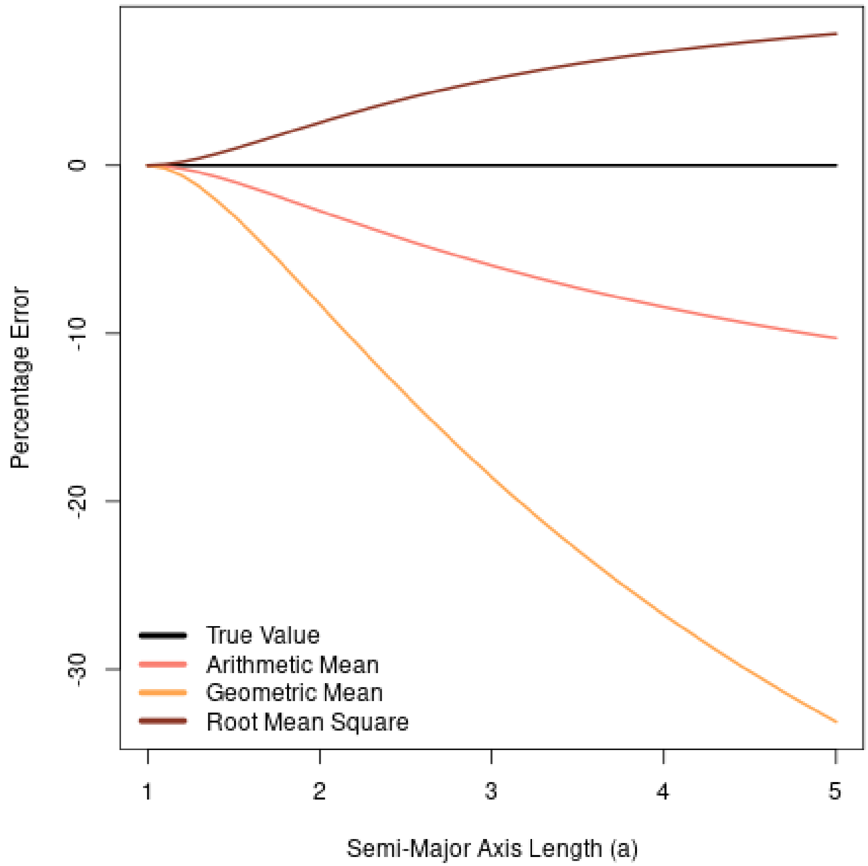
$$s = 2\pi\frac{a+b}{2}$$

(*Leonhard Euler, 1773*)

3. Root Mean Square

$$s = 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

(*Leonhard Euler, 1773*)



Exact Solutions

Colin Maclaurin (1742):

$$s = 2\pi a \sum_{m=0}^{\infty} \left(\left(\frac{(2m)!}{m!m!} \right)^2 \frac{k^{2m}}{16^m (1-2m)} \right) \quad \text{where } k = \sqrt{1 - \frac{b^2}{a^2}}$$

Leonhard Euler (1776):

$$s = \pi \sqrt{2(a^2+b^2)} \sum_{m=0}^{\infty} \left(\left(\frac{\delta}{16} \right)^m \cdot \frac{(4m-3)!!}{(m!)^2} \right) \quad \text{where } \delta = \left(\frac{a^2-b^2}{a^2+b^2} \right)^2$$

James Ivory (1796):

$$s = \pi(a+b) \sum_{m=0}^{\infty} \left(\left(\frac{(2m)!}{m!m!} \right)^2 \frac{h^m}{16^m (1-2m)^2} \right) \quad \text{where } h = \left(\frac{a-b}{a+b} \right)^2$$

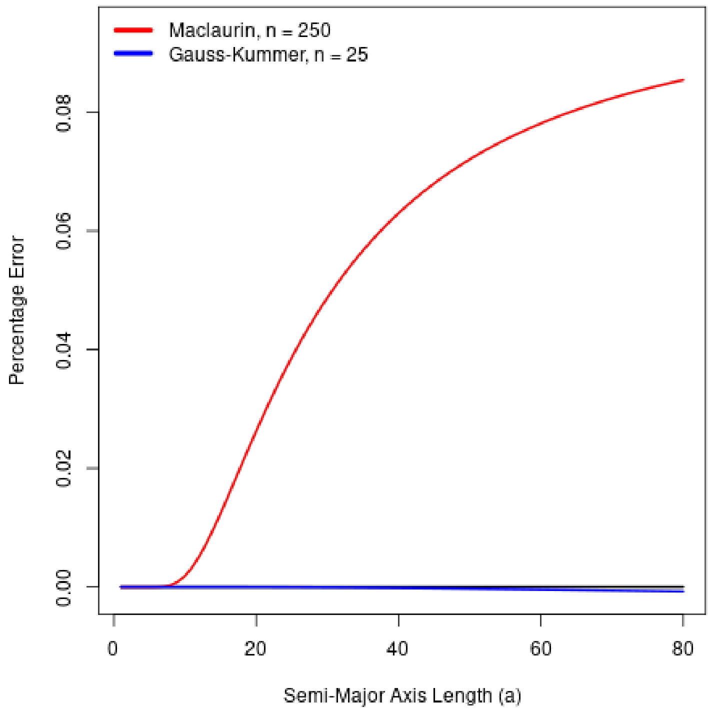
Carl Gauss (1812) & Eduard Kummer (1836) showed these could be expressed in terms of the hypergeometric function:

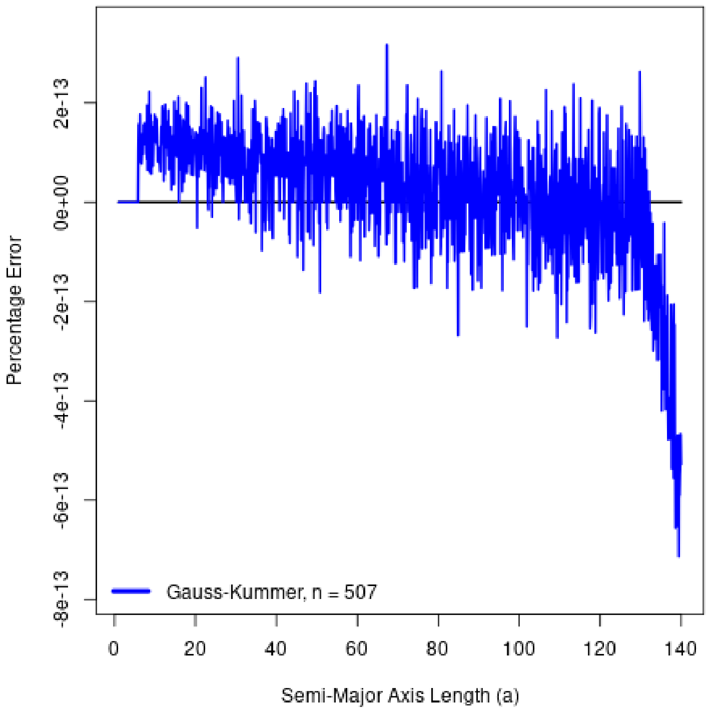
$$2\pi a \cdot F\left(-\frac{1}{2}, \frac{1}{2}, 1, k^2\right) = \pi \cdot F\left(-\frac{1}{4}, \frac{1}{4}, 1, \delta\right) = \pi(a+b) \cdot F\left(-\frac{1}{2}, -\frac{1}{2}, 1, h\right)$$

Approximating Exact Solutions

"Just use a power series to a high number of terms" Linden Disney-Hogg, 2021

Can sum first *n* terms but need to account for convergence rate:





Turns out James Ivory's series is actually really damn good.

Better Approximations

Srinivasa Ramanujan's Approximations (1914)

He gave two:

$$s = \pi\left(3(a+b) - \sqrt{(3a+b)(a+3b)}\right) \tag{1}$$

$$s = \pi(a+b)\left(1 + \frac{3h}{10+\sqrt{4-3h}}\right) \text{ where } h = \left(\frac{a-b}{a+b}\right)^2 \tag{2}$$

David Cantrell's Approximation (2001)

Dropped this equation on a Geometry Research Google group:

$$s = 4(a+b) - 2(4-\pi)\frac{ab}{H_p} \text{ where } H_p = \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}. \tag{3}$$

The value of p can be optimized for the type of ellipse:

$$p_{round} = \frac{3\pi-8}{8-2\pi}, \quad p_{long} = \frac{\ln(2)}{\ln(2/(4-\pi))}, \quad p_{general} = 0.825.$$

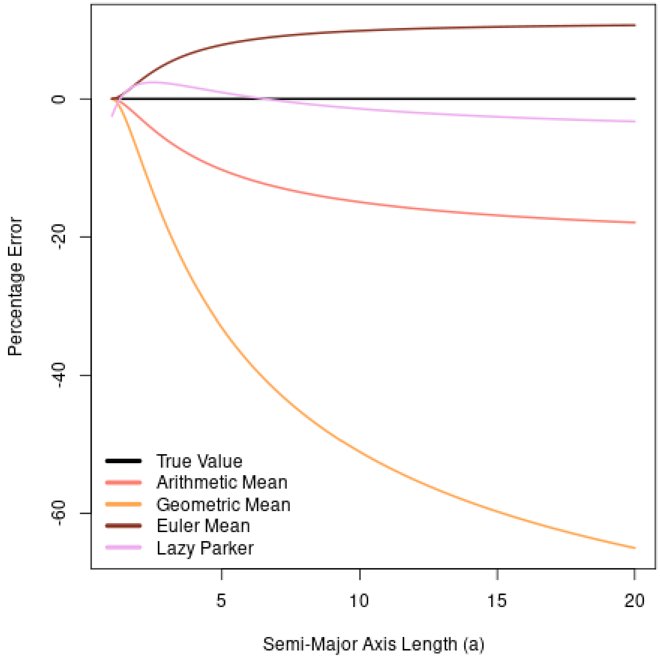
Better Approximations

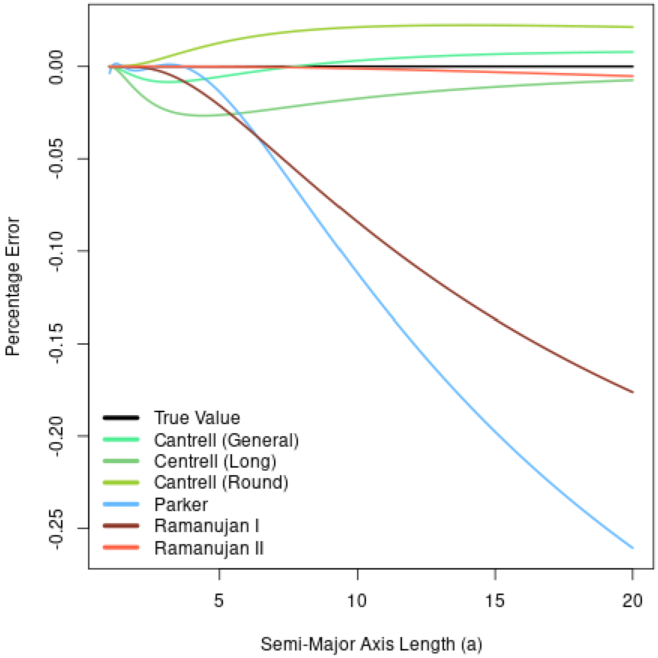
Parker's Approximation (2020)

$$s = \pi\left(\frac{53}{3}a + \frac{717}{36}b - \sqrt{269a^2 + 667ab + 371b^2}\right) \text{ where } a > b$$

Lazy Parker's Approximation (2020)

$$s = \pi\left(\frac{6}{5}a + \frac{3}{4}b\right) \text{ where } a > b$$





Monte Carlo: Idea

Monte Carlo for elliptic integrals first looked at by Gary Gipson (1982) as part of his PhD thesis, ‘*The Coupling of Monte Carlo Integration with the Boundary Integral Equation Technique to Solve Poisson Type Equations*’.

$$s = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^4 - a^2x^2}}\,\mathrm{d}x \quad (4)$$

$$s = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\,\mathrm{d}\theta. \quad (5)$$

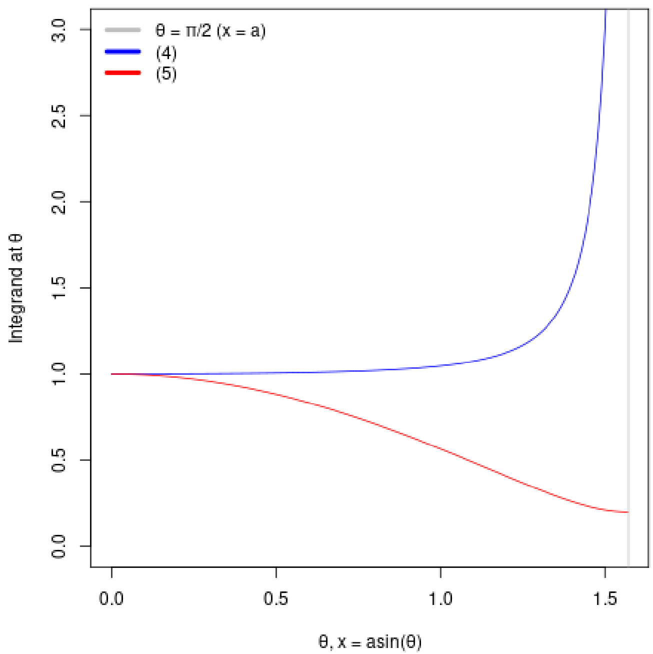
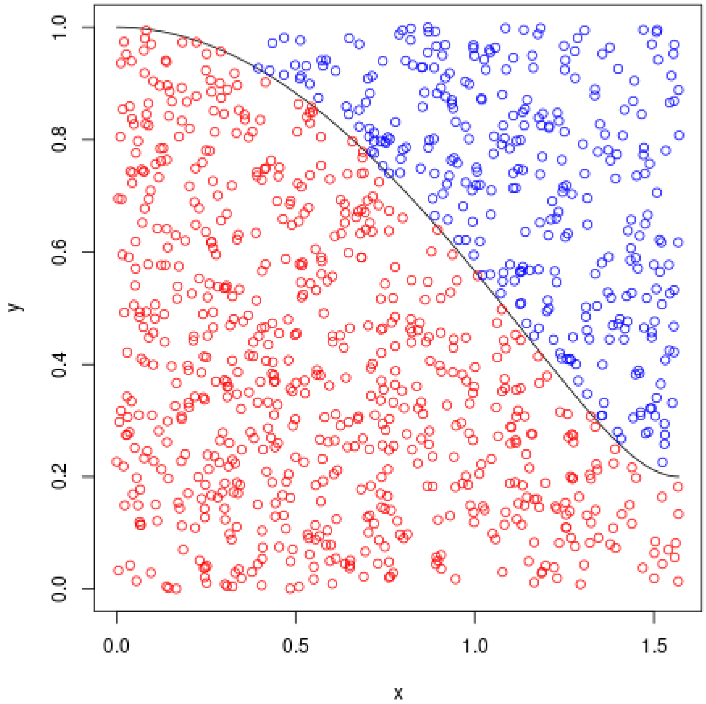
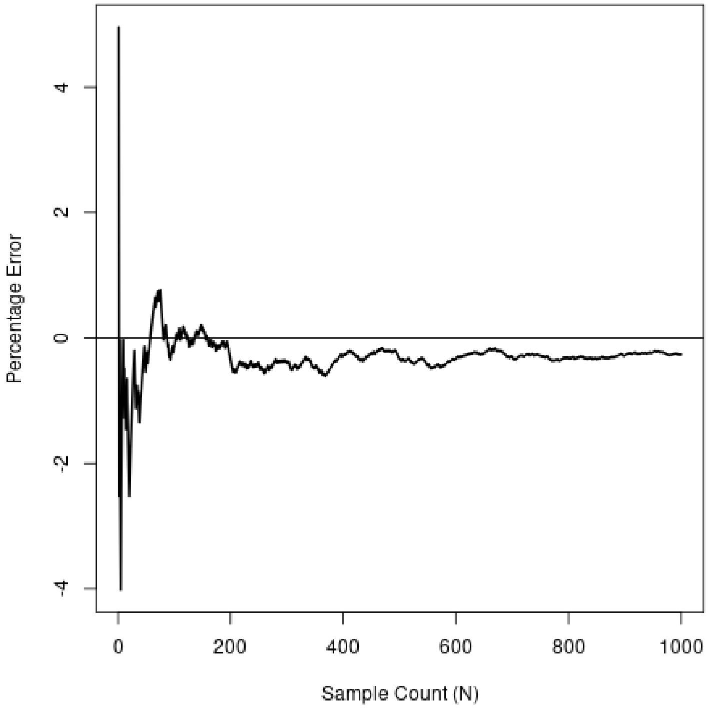


Figure: The two integrands we might try to use with Monte Carlo Integration.

Monte Carlo: Method

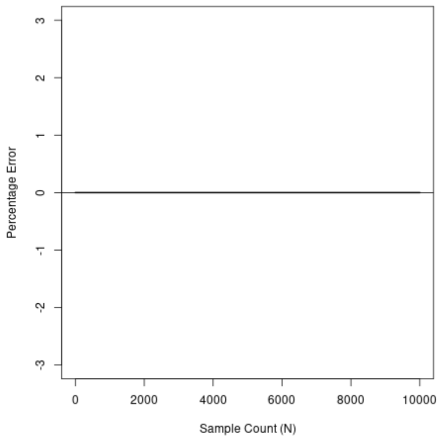
$$s \approx 4a \times \left(\frac{\pi}{2} \times 1\right) \times \frac{\#red}{\#red + \#blue},$$



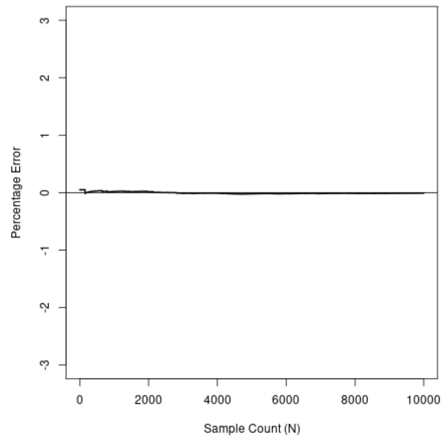


Monte Carlo: Convergence

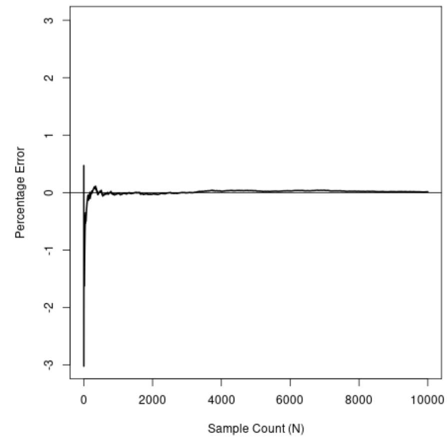
Plots of sample count against absolute percentage error for varying *a*.



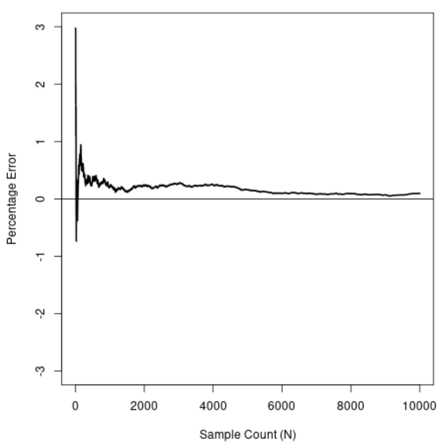
(a) $a = 1$



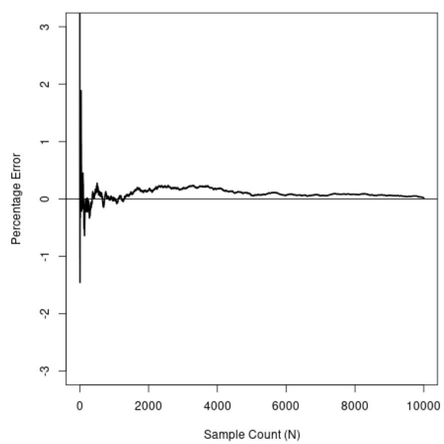
(b) $a = 1.01$



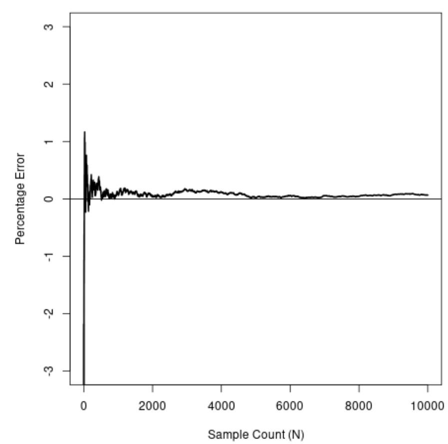
(c) $a = 1.1$



(d) $a = 2$



(e) $a = 5$



(f) $a = 80$

Comparing Them All

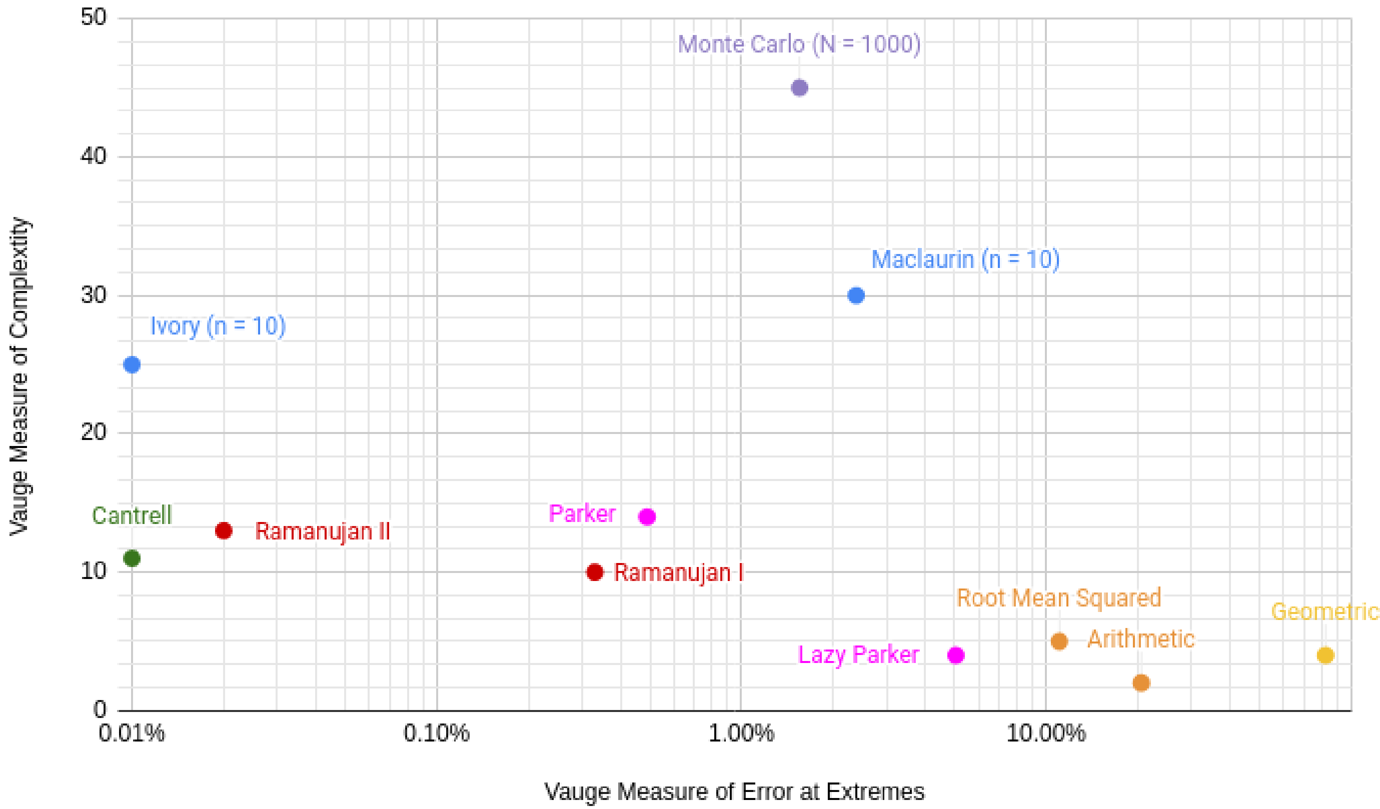


Figure: A vague comparison of the discussed methods based on complexity and accuracy.

Conclusion

1. Don't use Monte Carlo to do Elliptic Integrals in practice.
2. Like really don't...
3. ...but it *is* a cool exercise!

> *The theory of elliptic functions is the fairyland of mathematics. The mathematician who once gazes upon this enchanting and wondrous domain crowded with the most beautiful relations and concepts is forever captivated.* *-Richard Bellman*

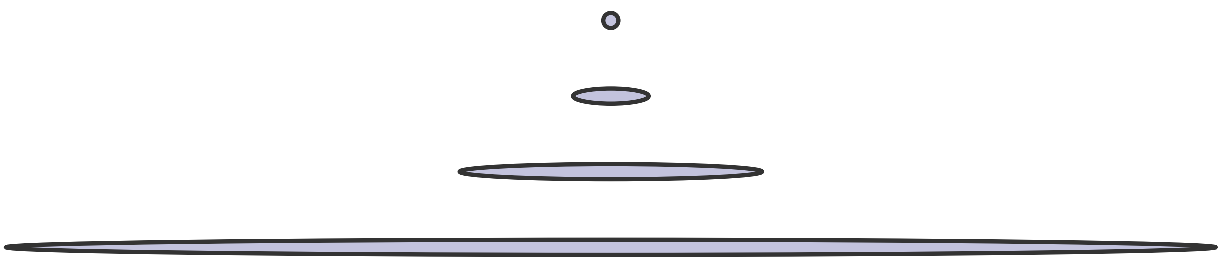


Figure: Ellipses with $b = 1$ and $a = 1, 5, 20$, and 80.

Figure: Ellipse with $b = 1$ and $a = 3.93$; has the same eccentricity as Halley's comet.

Figure: Ellipse with $b = 1$ and $a = 76.7$, has the eccentricity of most eccentric comet.