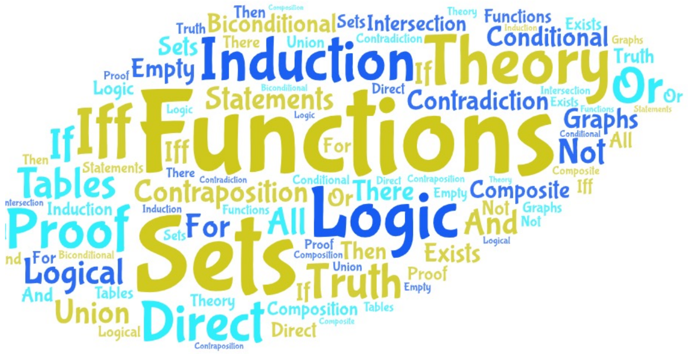
Discrete Mathematics and Probability (DMP): A new course

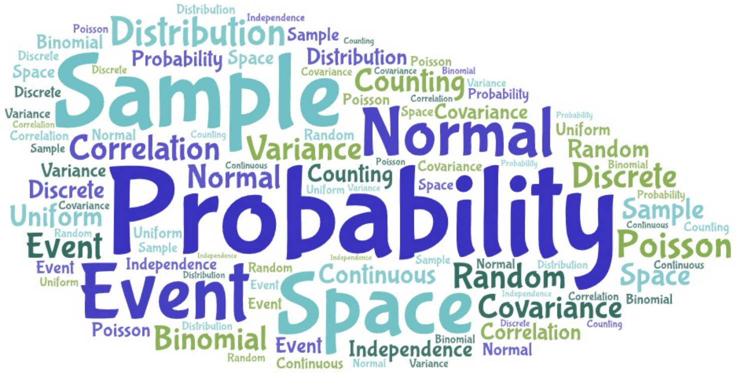
‘Replace a poor course by one resembling the way Mathematics run their popular courses’

DMP eventually replaced two 2nd year courses

(1) Discrete Mathematics and Mathematical Reasoning

(2) Probability with Applications





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Problems in previous Courses

- ~25% failed both!
- Very traditional courses - chalk and talk and a lot of student complaints
- Tutorial material was often too hard in Discrete Mathematics as it was written by experts in that field
- Aimed at stretching top students many of whom took Maths courses in Proof already in 1st year
- Probability course had double integrals and many students had not done Calculus since school/college
- Students did not engage each week with theory but left it to the last minute and hoped to pass exam

Process

- Watch previous lectures, get notes, tutor on DMMR
- Talk to current CO of DMMR
- Agree syllabus for both Discrete Maths and Probability
- Look at how it fits in detail with FDS and IADS
- Present at Teaching Committee then BoS
- Published to DPT

Central Problem!

How could we merge two very difficult courses into the space of 1?

Solution

Make sure that the foundation for Discrete Mathematics is laid in Inf1A

Note: Topics are touched on but no mathematical writing is done

Principles

- Service course – all students should engage with every part of theory

  Learning Outcomes

- Every part should be treated as equally useful and tested in some way

- Writing mathematics clearly is important

- Students should engage regularly and not leave learning until before an exam

- It should be enjoyable and include accessible book/notes, recorded videos as well as live sessions, peer instruction, peer and self-assessment, instructor friendliness

Learning Objectives

- Use mathematical and logical notation to define and formally reason about mathematical concepts such as sets, relations, functions, and integers, and discrete structures, including proof by induction.
- Use graph theoretic terminology and apply concepts from introductory graph theory to model and solve some basic problems in Informatics (e.g., network connectivity, etc.)
- Prove elementary arithmetic and algebraic properties of the integers, and modular arithmetic, explain some of their basic applications in Informatics, e.g., to cryptography
- Carry out practical computations with standard concepts from discrete and continuous probability, such as joint and conditional probabilities, expectations, variances, standardization.
- Recognize and work with standard discrete and continuous probability distributions and apply them to model and solve concrete problems.

Solution: stage 1

- Change textbook(s)
- Make tutorial material accessible with peer instruction & interactivity
- Regular marked homework with feedback and discussion in tutorials
- Two courseworks now Class Tests: Discrete Mathematics in Week 6, Probability in Week 12
- Weekly Quizzes

  Problem! Not many useful Discrete Mathematics questions in STACK

Adaptations for all online in 2020

- Moodle Book with notes and recorded videos (2020 so all online)
- Weekly structure
- Participation marks for forum & tutorials

  Too easy not to engage with content and be isolated

  ‘Excellent Online Teaching: Effective Strategies for a Successful Semester Online’ by Aaron Johnson

Quizzes in Stage 2

- First year we used the Publisher’s resources (best 8 out of 11)

  ‘The weekly quizzes and homeworks were really helpful because they gave us a reason to thoroughly look into each week’s material and they didn’t let us skip lectures and have to cram material just before courseworks, as was the cases with other courses.

  It was also really helpful that the weekly quizzes allowed us 3 tries for each question and weren’t timed, because we didn’t have to stress about the marks and we could actually figure out how to do each exercise.’

  ‘ The quiz questions were often too simple and mundane’

- Second year in Summer 2021 hired an intern (just graduated student) in joint Computing and Maths degree to write a bank of STACK resources

- Thanks to Adrian Doña Mateo who did an amazing job!

Compound statements and Logic Equivalence

Which one of the following propositions corresponds to the statement

'Eva will attend the party if Adam won't, but if Adam attends the party then Bertha won't attend',

if we denote:

- $A =$ 'Adam will attend the party'
- $B =$ 'Bertha will attend the party'
- $E =$ 'Eva will attend the party'

Select one:

a. $(\neg A \rightarrow E) \wedge (A \rightarrow \neg B)$

b. $(E \rightarrow \neg A) \wedge (A \rightarrow \neg B)$

c. $(E \vee \neg A) \rightarrow (A \vee \neg B)$

d. $E \rightarrow (A \rightarrow \neg B)$

e. $A \rightarrow (E \wedge \neg B)$

Block 1: Discrete Mathematics

Build up in sections for scaffolding:

Properties and

- Proof with elementary number theory (integers then rationals)
- Proof with sequences (& Mathematical Induction and Recursion)
- Proof with set theory,
- Proof with functions
- Proof with relations
- (Proof with Graph Theory)

Methods of Proof

- Within proof with elementary number theory they meet in order
    a) Direct proof and counterexample
    b) Proof by cases
    c) Contraposition
    d) Contradiction

- Mathematical Induction is taught separately then Strong Mathematical Induction and Well-Ordered Principle for integers

- Students are told if they have to use induction then it will say in the question, otherwise they are free to use in other questions, but it is unlikely to be the easiest method.

Proof with 'drag and drop' questions

Let m and n be positive integers. Consider the following statement:

'If mn is a perfect square, then m and n are perfect squares.'

Construct the converse, the contrapositive and the negation of this statement.

- Converse:

If [ ], then [ ].

- Contrapositive:

If [ ], then [ ].

- Negation:

There exist [ ], and yet [ ].

For each of these, select whether they are true or false.

- The converse is [ ].
- The contrapositive is [ ].
- The negation is [ ].

m and *n* are perfect squares	*mn* is a perfect square	*mn* is not a perfect square
neither *m* nor *n* is a perfect square		

at least one of *m* and *n* is not a perfect square	*mn* is not a perfect square	neither *m* nor *n* is a perfect square
mn is a perfect square		

m and *n* not both perfect squares	*mn* is a perfect square	*m* and *n* both not perfect squares	*mn* is not a perfect square

true	false
false	true

Thorough feedback

: m and n be positive integers and P, M and N be the propositions:

$$P = \text{'}mn \text{ is a perfect square'}$$
$$M = \text{'}m \text{ is a perfect square'}$$
$$N = \text{'}n \text{ is a perfect square'}$$

e given statement can then be written as $P \to (M \wedge N)$. Using symbols will make our task of finding the converse, contrapositive and negation easier.

The converse of $P \to (M \wedge N)$ is simply $(M \wedge N) \to P$. In English, this can be written as

'If m and n are perfect squares, then mn is a perfect square'.

This statement is true, since if we have $m = k^2$ and $n = l^2$ then $mn = k^2 l^2 = (kl)^2$, which is a perfect square.

The contrapositive of $P \to (M \wedge N)$ is $\neg(M \wedge N) \to \neg P$. Using de Morgan's laws, this can be simplified as $(\neg M \vee \neg N) \to \neg P$. In English, this can be written as

'If at least one of m and n is not a perfect square, then mn is not a perfect square'.

This statement is false. For example, if we take $m = n = 2$ then neither is a perfect square, ans still $mn = 2^2$ is a perfect square.

The negation of $P \to (M \wedge N)$ can be found by first rewriting the implication as a disjuction: $\neg P \vee (M \wedge N)$. The negation is then

$$\neg(\neg P \vee (M \wedge N)) \equiv P \wedge \neg(M \wedge N) \equiv P \wedge (\neg M \vee \neg N),$$

were we also used de Morgan's laws.

This can be written in English as

'There exist m and n not both perfect squares, and yet mn is a perfect square'.

This is statement is true. We can take the same example as above, that is, $m = n = 2$ and $mn = 2^2$.

Proof with 'drag and drop' questions

Drag the phrases below into the spaces provided to form the negations of these statements.

A. For all integers m, there exists an integer n such that mn is odd.

.

B. There exists an integer m such that, for all integers n, mn is even.

.

Now select whether each orginal statement is true or false.

Statement A is [ ] .

Statement B is [ ] .

There exists an integer m such that	for all integers n	mn is even
For all integers m	there exists an integer n such that	mn is odd

false | true

Proof with 'drag and drop' question feedback

A. 'For all integers m, there exists an integer n such that mn is odd' translates to:

$$\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, mn \text{ is odd.}$$

Now we form the negation. Recall that when negating a quatified statement we must swap the quatifiers.

$$\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, mn \text{ is even.}$$

Written in English, this reads as 'There exists an integer m such that, for all integers n, mn is even'.

B. 'There exists and integer m such that, for all integers n, mn is even' translates to:

$$\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, mn \text{ is even.}$$

We negate the symbolic expression, taking care of the quatifiers.

$$\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, mn \text{ is odd.}$$

Written in English, this reads as 'For all integers m, there exists an integer n such that mn is odd'.

Finally, note that statements A and B are the negation of each other. This means that one of them must be true and the other must be false.

In this case, B is true. To see this, one can take $m = 2$. Then no matter what n is, $mn = 2n$ will always be even.

The correct answer is:
Drag the phrases below into the spaces provided to form the negations of these statements.

A. For all integers m, there exists an integer n such that mn is odd.

[There exists an integer *m* such that] [for all integers *n*] [*mn* is even].

B. There exists an integer m such that, for all integers n, mn is even.

[For all integers *m*] [there exists an integer *n* such that] [*mn* is odd].

Proof reading comprehension

Let a and b be integers. Which of the following proofs establishes the following statement?

'If ab is odd, then a and b are both odd.'

A. Suppose a and b are odd. Then there are some integers k and m such that $a = 2k + 1$ and $b = 2m + 1$, and we have

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1, \end{aligned}$$

which is odd.

B. Suppose a and b are even. Then there are some integers k and m such that $a = 2k$ and $b = 2m$, and we have

$$ab = 2k \cdot 2m = 2(2km),$$

which is even.

This shows that, if a and b are both even, then ab is even. The result follows by taking the contrapositive.

C. Suppose one of a and b is even. First, say a is even. Then there is an integer k such that $a = 2k$ and $ab = 2k \cdot b = 2(kb)$, which is even. Similarly, if b is even, then there is an integer m such that $b = 2m$ and $ab = a \cdot 2m = 2(am)$, which is even.

This shows that, as soon as one of a and b is even, then ab is even. The result follows by taking the contrapositive.

Which proofs are correct?

Proof by Cases

Tidy STACK question tool | Question tests & deployed va

In this question you will prove that, for any integer n, the integer n^2 is a multiple of 3 *if and only if* n is itself a multiple of 3. Fill in the blanks to complete the proof.

Let n be any integer. By the quotient-remainder theorem, the remainder of n upon division by 3 must be either 0, 1 or 2. We study each case separately:

1. First, suppose said remainder is 0. Then n is a multiple of 3 and there exists some integer k such that $n =$ (No answer given) .

   Squaring both sides we see that

   $$n^2 = \quad ,$$

   which is (No answer given) .

2. Now, suppose that the remainder is 1. Then there exists some integer k such that $n =$ (No answer given) .

   Squaring both sides we see that

   $$n^2 = \quad ,$$

   which is (No answer given) .

3. Lastly, suppose that the remainder is 2. Then there exists some integer k such that $n =$ (No answer given) .

   Squaring both sides we see that

   $$n^2 = \quad ,$$

   which is (No answer given) .

Which cases show that 'if n is a multiple of 3 then so is n^2'? (No answer given)

Which cases show that 'if n^2 is a multiple of 3 then so is n'? (No answer given)

You may need to take the contrapositive of the relevant statement to answer these questions.

Set theory questions

Let $A = \{2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{3, 5, 7, 9\}$ be sets.

Find each of the following sets. Write sets using curly braces and comma-separated elements, for example write `{1,2,3}` for $\{1, 2, 3\}$.

a. $(A \cup B) \cap C =$

b. $(A \cap B) \cup C =$

c. $A \cup B \cup C =$

Tidy STACK question tool | Question tests & deployed variants

Let $A = \{3, 5, 9, 10\}$ and $B = \{0, 2, 4, 6, 8, 10\}$ be sets, and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set.

Recall that the complement of a set X is written by X^C.

Find each of the following sets. Write sets using curly braces and comma-separated elements, for example write `{1,2,3}` for $\{1, 2, 3\}$.

a. $A^C =$

b. $B^C =$

c. $A \cup B =$

d. $(A^C \cap B)^C =$

e. $(B^C \cup A)^C =$

Functions

Tidy STACK question tool | Question tests & deployed

Given $f(x) = x - 8$ and $g(x) = 6x + 2$ find the composite functions:

a. $(g \circ f)(x) =$

b. $(f \circ g)(x) =$

Give your answers in expanded form.

Let $X = \{1, 2, 3, 4\}$ and consider the following functions $f, g : X \to X$ (from the set X to itself):

$$f = \{[1,4], [2,3], [3,1], [4,3]\}$$
$$g = \{[1,3], [2,4], [3,1], [4,2]\}$$

These are given formally as subsets of the cartesian product $X \times X$. Pairs in $X \times X$ are written with square brackets in this case.

Determine the compositions $f \circ g$ and $g \circ f$. Give your answers as subsets of $X \times X$, with pairs written in square brackets, like f and g above.

$f \circ g =$

$g \circ f =$

Properties of Relations: basics

Consider the following relations on the natural numbers. For each of them, select what properties it satisfies.

1. The divisbility relation: $x \mid y$.

   ☐ reflexive ☐ symmetric ☐ transitive

2. The greater than relation: $x > y$.

   ☐ reflexive ☐ symmetric ☐ transitive

3. The non-equality relation: $x \neq y$.

   ☐ reflexive ☐ symmetric ☐ transitive

Tidy STACK question

Let S be the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. Consider the relation R on S given by

$$x\,R\,y \iff 4 \mid (x^2 - y^2).$$

It is a fact that R is an equivalence relation (can you see why?).

Find the partition of S given by the equivalence classes of R. Write your answer as a set of sets, for example `{{-4,-3}, {0}, ...}`.

Properties of Relations: Congruence Equations needed for IADS

Tid

In this question, you will learn to solve simultaneous congruence equations of the form

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases},$$

where m and n are coprime positive integers.

Let us examing the following example:

$$\begin{cases} x \equiv 6 \pmod{7} \\ x \equiv 10 \pmod{13} \end{cases}$$

First, since 7 and 13 are coprime, Bézout's identity tells us that there exists integers r and s such that

$$7r + 13s = 1.$$

Find values for r and s using the Euclidean algorithm.

$r =$ ____ $s =$ ____

Note that, if we have $7r + 13s = 1$, then

$$13s \equiv 7r + 13s = 1 \pmod{7}$$

and

$$7r \equiv 7r + 13s = 1 \pmod{13}.$$

Now, for some integers a and b, consider the number $7br + 13as$. What does this number reduce to modulo 7 and 13?

$7br + 13as \equiv$ ____ $\pmod{7}$

$7br + 13as \equiv$ ____ $\pmod{13}$

Selecting appropriate values for a and b, find a solution x to the system (1) of congruence equations.

$x =$ ____

Block 2: Probability needed for FDS

- Counting techniques: product rule, permutations, combinations
- Axioms of probability, sample space, events, De Morgan's Law
- Joint and conditional probability, independence, chain rule, law of total probability, Bayes' Theorem
- Random variables, expectation, variance, covariance
- Common discrete and continuous distributions (e.g., Bernoulli, binomial, Poisson, uniform, exponential, normal)
  - Central limit Theorem
- Emphasis on practical applications of probability

Regular Homework

- Emphasis on being able to write mathematics clearly
- Accuracy, legibility, systematic development from line to line
- Encourage peer assessment

Ensuring that the proof was methodical and there were no gaps or no missing explanations in any of the logical steps.

I have learnt from others' solution of 1b) and there are many ways to do it.

a minor silly mistake where i had written the wrong value but referred to the correct one later in the proof - need to make sure that i check my work over more carefully next time

My proof is quite messy, I wrote 5 as 3

After a bit of nitpicking, some of the members forgot to define the variables the detail
For example, when proving n^2 is even many students wrote n^2 = (2k)^2 but forgot to define k as an integer

I learned that it is especially important to explain explicitly every passage of the proof, so that both I and the reader don't lose track of how the proof proceeds. There were quite a few oversights that I could have avoided if I had not completed the homework close to the deadline

Q1: Since $P \to Q \equiv \neg Q \to \neg P$

so the contraposition will be:

if $m \geq 10$ and $n \geq 12$, then $mn \geq 120$

if $m \geq 10$

then $mn \geq 10n$

because $n \geq 12$

then $10n \geq 120$

then $mn \geq 10n \geq 120$

So we have $mn \geq 120$

So we have proved if $m \geq 10$, $n \geq 12$, then $mn \geq 120$.

So the contraposition is True, then this statement is True.

the contrapositive of $mn < 120 \to (m<10 \vee n<12)$ is:

$$\neg(m<10 \vee n<12) \to \neg(mn<120)$$

$$\therefore \quad (m \geq 10 \wedge n \geq 12) \to mn \geq 120$$

and using the Assumption "if $a \geq b$ and $c \geq d$ then $ac \geq bd$" we can say as $m \geq 10$ & $n \geq 12$, $10 \times 12 = 120$

$\therefore mn \geq 120$

(for $a,b,c,d \in \mathbb{N}$ & $c>0$)

So the contrapositive is true following the assumption. & as the contrapositive is proven true we have proven also that the original statements is True.

1. Mn< 120 then m <10 or n <12 therefore the contraposition is

Mn>=120 if m=> 10 and n=>12

Take m = 10, n = 12. It’s true that mn =120 since 10*12=120. therefore for every number greater this is true so by contraposition mn<120 if m<10 or n<12

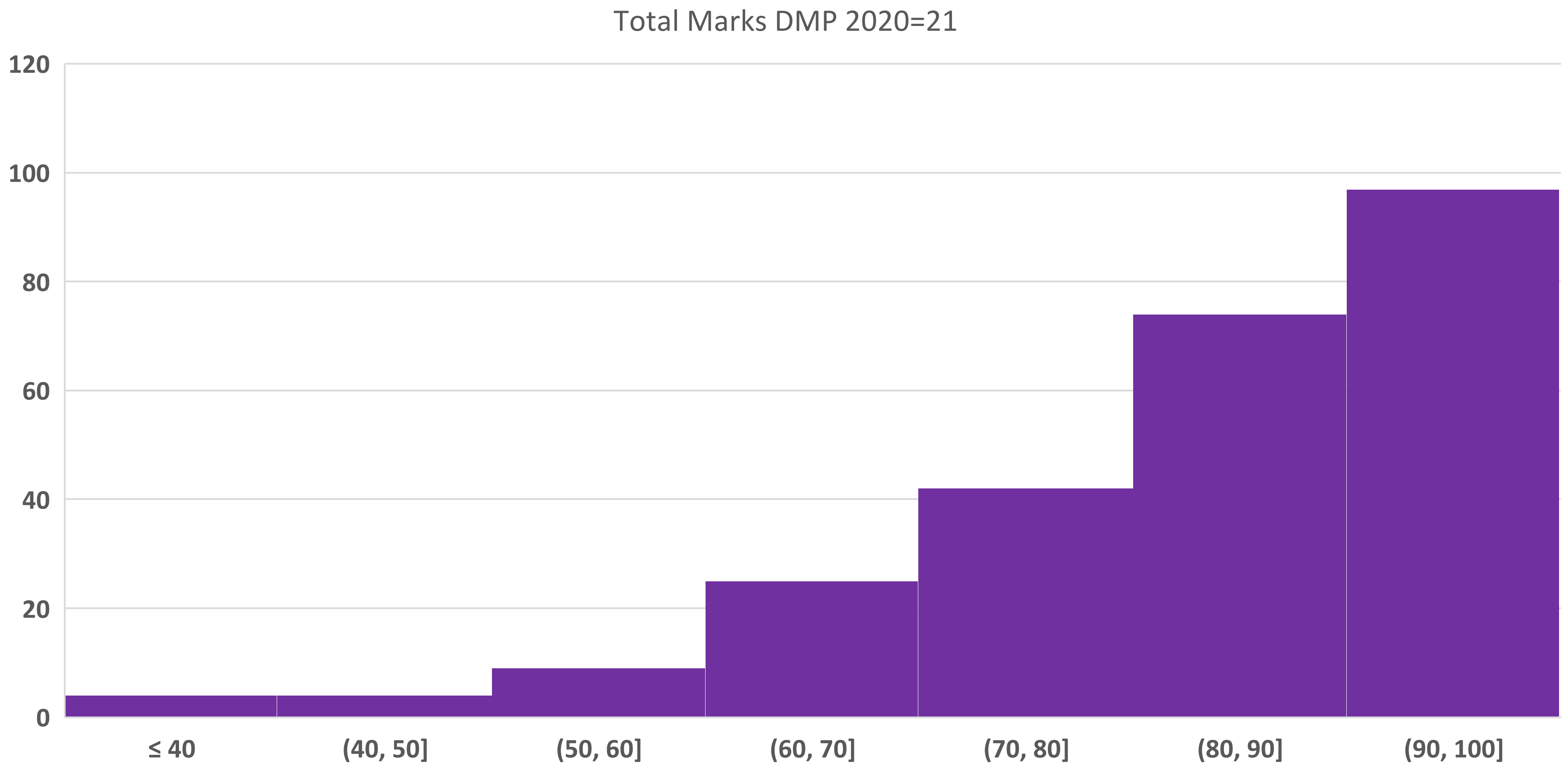
$m \in \mathbb{Z}$ $n \in \mathbb{Z}$ prove by contraposition
$mn < 120$ then $m < 10$ $n < 12$

assuming $a \geq b$ $c \geq d$ $ac \geq bd$

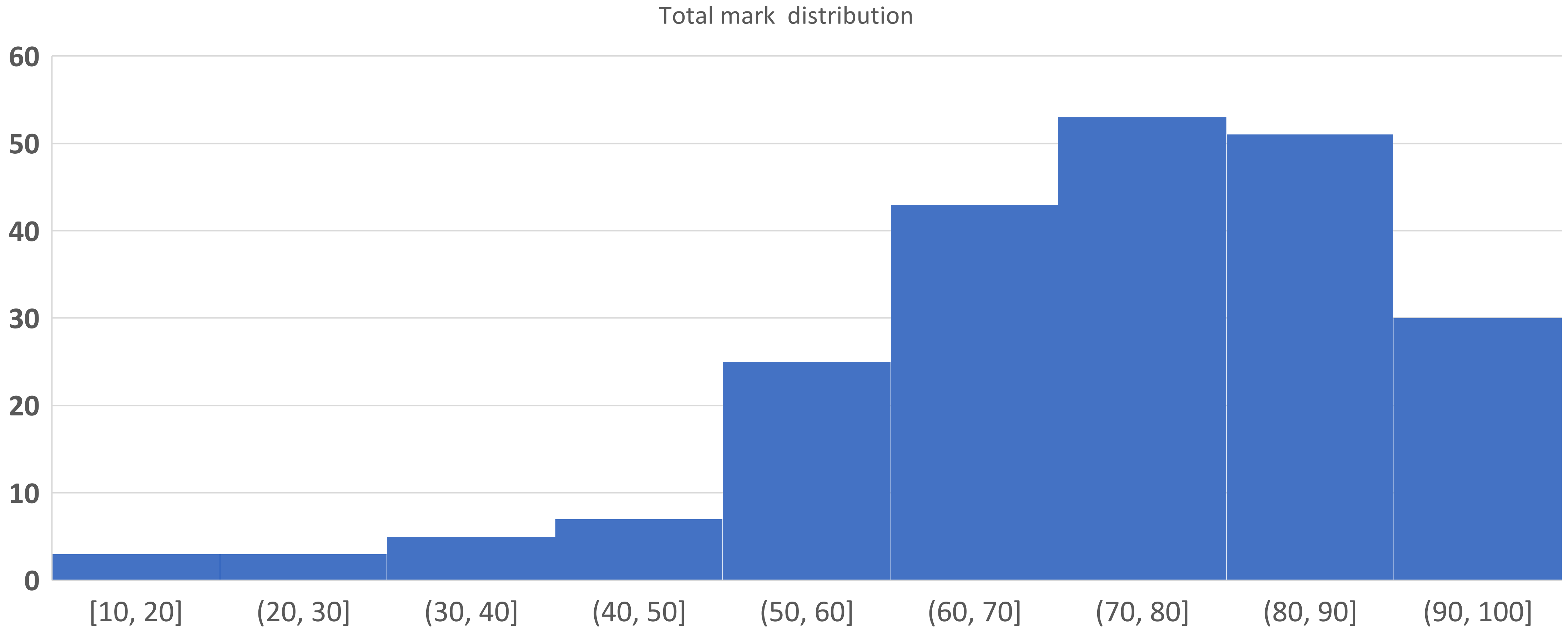
$\therefore$ if $mn \geq 120$ $m \geq 10$ and $n \geq 12$

as $a > b$, $c > d$ $ac > bd$
$m \geq 10$, $n \geq 12$ $mn \geq 120$ $\therefore$ the opposite is
$\cancel{m > 0}$
also true

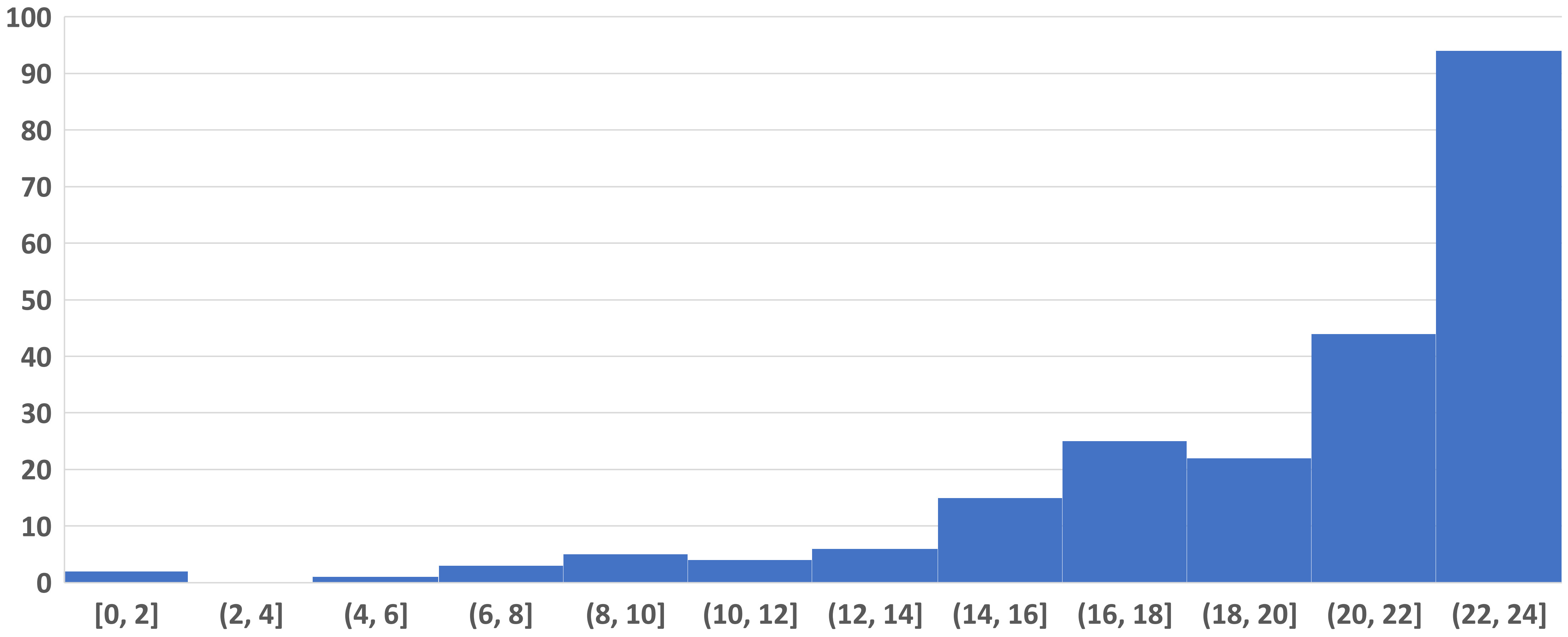
DMP Number of students and marks 2020-21



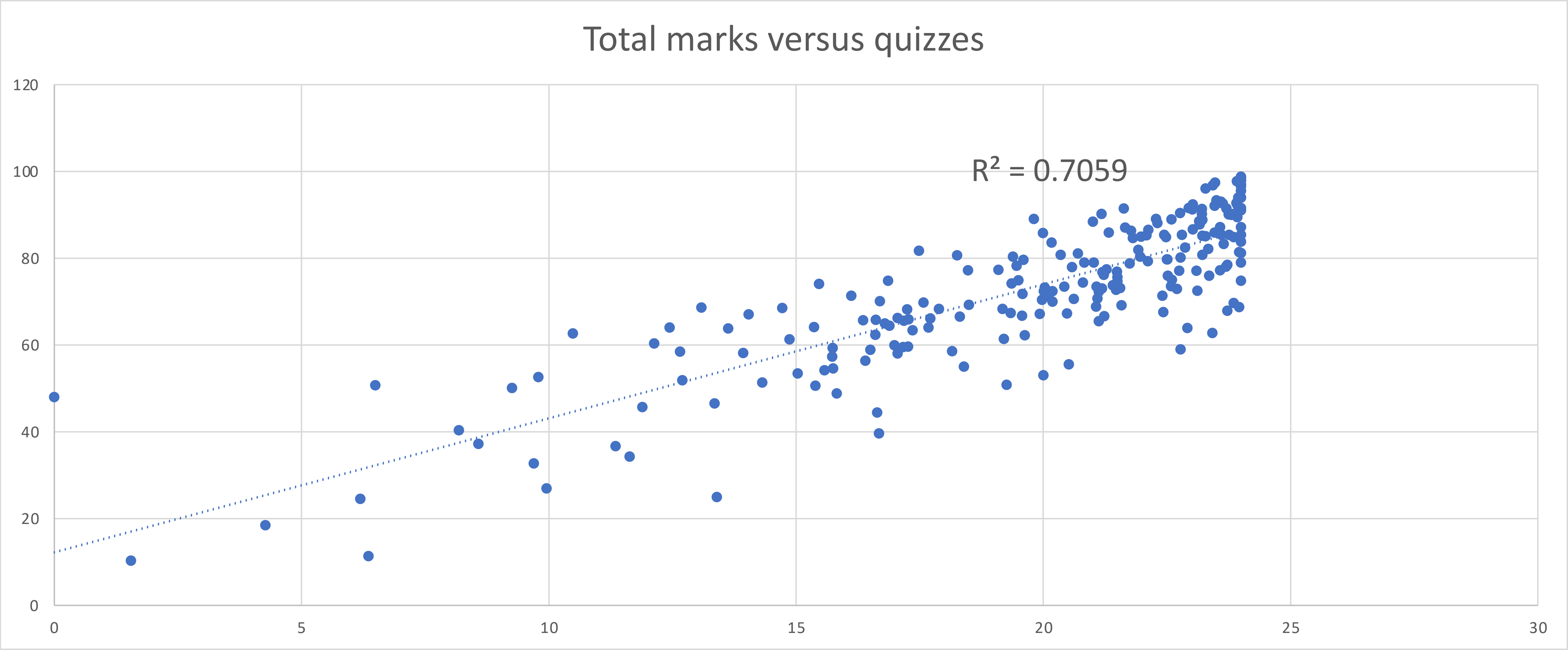
DMP Number of students and marks 2021-22



Quiz: Number of Students and Mark Distribution out of 24%



2021-22 Quiz Marks versus Final Mark



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