Reciprocals

Theorem Let $(a_n) \to a$ and suppose that $a \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$. Then

$$\left(\frac{1}{a_n}\right) \to \frac{1}{a}.$$

Proof

(L1) Since $(a_n) \to a$ there exists N_1 such that $|a_n - a| < \frac{|a|}{2}$ for all $n > N_1$.

(L2) Then, for $n > N_1$ we have $|a_n| > \frac{|a|}{2}$.

(L3) Now let $\varepsilon > 0$ be given.

(L4) Let N_2 be such that $|a_n - a| < \frac{\varepsilon |a|^2}{2}$ for all $n > N_2$.

(L5) Let $N = \max\{N_1, N_2\}$.

(L6) Then for $n > N$ we have $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|}$

(L7) $< \frac{|a_n - a|}{|a|^2/2}$

(L8) $< \frac{\varepsilon |a|^2/2}{|a|^2/2}$

(L9) $= \varepsilon.$

(L10) Thus $\left(\frac{1}{a_n}\right) \to \frac{1}{a}$ as claimed.

1. Which statement best expresses the meaning of Line 2?

   (a) Beyond a certain term in the sequence, every term has absolute value greater than a certain constant.

   (b) There exist infinitely many terms in the sequence which have absolute value greater than a certain constant.

   (c) The absolute value of the terms in the sequence is always greater than one half the absolute value of the limit.

2. Which of the following statements is closest in meaning to Line 3?

   (a) Since $(a_n) \to a$ from the statement of the theorem, the definition of convergence gives a positive number ε which we now consider.

   (b) Now we shall assume that we are given ε which may be any positive number.

   (c) Now we are introducing a new variable ε, the value of which is positive and which will be determined in later calculations.

3. Would the proof still be correct if in Line 5 we replaced "$N = \max\{N_1, N_2\}$" with "$N = N_1 + N_2$"?

   (a) No because it does not make sense to add N_1 and N_2.

   (b) No because $N_1 + N_2$ is larger than $\max\{N_1, N_2\}$.

   (c) Yes because it would still be the case that $n > N$ implies $n > N_1$ and $n > N_2$.

4. Where does the expression $\frac{\varepsilon|a|^2}{2}$ on the right-hand side of the inequality in Line 4 come from?

   (a) It comes from the inequality in Line 1, observing that $|a|^2 \geqslant |a|$ and ε is arbitrary.

   (b) It has been chosen so that the calculations in Lines 6–9 work.

   (c) It has been chosen so that ε cannot be too close to zero for the given value of N_2.

5. Where is the result in Line 2 used later in the proof?

   (a) It is needed so that we know ε in line 3 can be chosen to be positive.

   (b) It is needed to deduce Line 7 from Line 6.

   (c) It is needed to deduce Line 8 from Line 7.

6. How do we know in Line 1 that such an N_1 exists?

   (a) Because the value of ε in Line 3 has not yet been fixed.

   (b) Because $(a_n) \to a$, the terms of the sequence must eventually be within $|a|/2$ of a.

   (c) Because it is the start of the proof we can assume anything we need.

7. Which of the following best captures the content of the proof?

   (a) Given positive ε, there is a term in the sequence beyond which $1/a_n$ is always within distance ε of $1/a$.

   (b) There is a number N such that when $n > N$ we have $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon$ for all $\varepsilon > 0$.

   (c) We can find a natural number N and $\varepsilon > 0$ such that when $n > N$ we have $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon$.

8. How do you expect the values needed for N_1, N_2 would normally change as ε becomes smaller?

   (a) Both N_1 and N_2 will become larger.

   (b) N_1 will become larger but N_2 can stay the same.

   (c) N_2 will become larger but N_1 can stay the same.

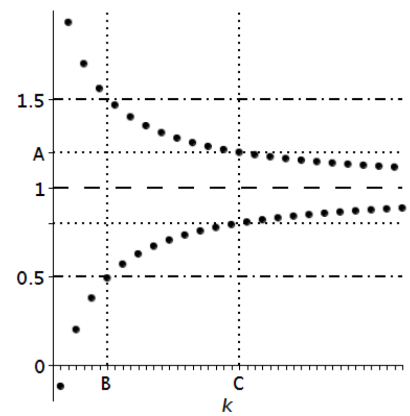
9. Suppose that $b \neq 0$ and $b_n \neq 0$ for all n and that $(1/b_n) \to 1/b$. Can we use the Theorem to deduce that $(b_n) \to b$?

   (a) No the theorem only tells us that if $(b_n) \to b$ then $(1/b_n) \to 1/b$. (It is not an "if and only if" theorem.)

   (b) Yes, apply the theorem with $a_n = 1/b_n$ and $a = 1/b$.

   (c) No, because the algebra in the proof between Line 6 and Line 8 will not work in this case.

10. The picture shows the first 45 terms of a possible sequence (a_n) converging to $a = 1$.



    Which labeling of the points A,B,C marked on the axes would illustrate the proof?

   (a) A is $1 + \varepsilon$, B is N_1 and C is N_2.

   (b) A is $1 + \varepsilon/2$, B is N_2 and C is N_1.

   (c) A is $1 + \varepsilon/2$, B is N_1 and C is N_2.