Nested intervals

Theorem Let $I_k = [a_k, b_k]$ for $k \in \mathbb{N}$ be a collection of closed intervals with $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$. Then the intersection $X = \bigcap_{k=1}^{\infty} I_k$ is non-empty.
If also the width of the intervals I_k tends to zero as $k \to \infty$ then X is a single point.

Proof

(L1) Observe that for all k and l we have $a_k \leqslant b_l$ because if there were k and l such that $a_k > b_l$ then we would not have one of I_k and I_l contained in the other.

(L2) The sequence (a_k) is non-decreasing and is bounded above by b_1 and so $(a_k) \to a$ for some a.

(L3) For fixed l, we have $a_k \leqslant b_l$ for all k

(L4) and so the limit a must satisfy $a \leqslant b_l$.

(L5) Since also $a \geqslant a_l$ for all l we have that a is in all of the intervals I_l.

(L6) Hence $a \in X$, so X is non-empty.

(L7) Suppose now that also the width of the intervals I_k tends to zero and let $c \in X$.

(L8) If $c \neq a$, then for k sufficiently large we have $b_k - a_k < |c - a|$.

(L9) But for such k we could not have both c and a in I_k, which contradicts the fact that both $a \in X$ and $c \in X$.

(L10) Hence $c = a$ and the intersection is a single point.

1. In the situation of the theorem, how are I_{10} and I_{20} related?

   (a) We know only that the intersection of I_{10} and I_{20} is non-empty.

   (b) It must be the case that I_{20} is a subset of I_{10}.

   (c) It must be the case that I_{10} is a subset of I_{20}.

2. The statement $x \in \bigcap_{k=1}^{\infty} I_k$ means which of the following?

   (a) That x is in I_k for all sufficiently large k.

   (b) That x is in all of the sets $I_1, I_2, I_3, \ldots$

   (c) There exists a number N such that $x \in I_1 \cap I_2 \cap \cdots \cap I_N$.

3. In Line 7, why do we introduce an element named c when we know $c = a$?

   (a) We want to show that a is the only element of X and so we consider a possibly different element c of X and prove that $c = a$.

   (b) Because if (a_k) does not converge, it may be that a is not in X.

   (c) Because we are now making an extra assumption on the intervals I_k, that their width tends to zero.

4. How do we know in Line 5 that $a \geqslant a_l$ for all l?

   (a) By the squeeze theorem, because the width of the intervals I_k tends to zero.

   (b) Because a is in all of the intervals I_k.

   (c) Because (a_k) is a non-decreasing sequence and a is its limit.

5. In the deduction in Line 1, how do we know that if $a_k > b_l$ we could not have one of I_k and I_l contained in the other?

   (a) Because the sequence (a_k) is non-decreasing and the sequence (b_k) is non-increasing.

   (b) Because $a_k > b_l$ would mean that the left-hand end of I_k was greater than the right-hand end of I_l.

   (c) Because if $a_k \leqslant b_l$ then there would be a point x such that $a_k \leqslant x \leqslant b_l$ that is in both the intervals.

6. Which of the following best summarises the argument in lines 2–6?

   (a) We deduce that $a_k \leqslant b_l$ for all k, l and so conclude that a point that is in all the intervals I_k must be the limit of the sequence (a_k).

   (b) We assume that we have a point a that is in all the intervals I_k and deduce that $a \geqslant a_k$ for all k and so it is the limit of the sequence (a_k).

   (c) We consider the sequence whose terms are the left-hand ends of the intervals I_k and show that this sequence converges and that its limit is in all the intervals.

7. The condition that the widths of the intervals tends to zero as k tends to infinity is being assumed in which parts of the proof?

   (a) Line 7 only.

   (b) Lines 7–10 only.

   (c) The whole proof except for Line 1.

8. Could an argument very similar to Lines 7–10 of the proof be used to establish the following claim?

   > **Claim:** Let $I_k = (a_k, b_k)$ for $k \in \mathbb{N}$ be a collection of open intervals such that the width of the intervals I_k tends to zero as $k \to \infty$. If the intersection $\bigcap_{k=1}^{\infty} I_k$ is non-empty then it consists of a single point.

   (a) Yes: let a and c be in the intersection and proceed as in lines 7–10.

   (b) No: we need that the intervals satisfy $I_{k+1} \subseteq I_k$ so that the "sufficiently large" k in Line 8 is certain to exist.

   (c) No: we need that the intervals I_k are closed otherwise we may have $a \notin I_k$ for some k.

9. Could the proof be made to work instead by considering the sequence (b_k) and showing that its limit b is in the intersection X?

   (a) No, because we have proved that it is the limit a of the sequence (a_k) which is in X and it may be that $a \neq b$.

   (b) No, because then we would need to have $I_k \subseteq I_{k+1}$ instead of $I_{k+1} \subseteq I_k$.

   (c) Yes, (b_k) is non-increasing and bounded below and so has a limit b that we could prove to be in X in an analogous way.

10. Suppose the intervals I_k in the theorem are given by

$$I_k = \left[\frac{k-1}{2k}, \frac{k+1}{k}\right].$$

   Then according to the method of the proof we have

   (a) $a = 1/2$

   (b) $a = 3/4$

   (c) $a = 1$